KUKA youBot

Stanford Robotic Platforms

KUKA/DLR Lightweight Robot IV

CS225A 2014-Projects

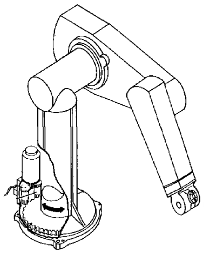
- PUMA (two arms) – 4 groups
- KUKA Lightweight Robot – 2 groups
- KUKA youBot Robot – 1 group
- Haptic Simulation – 3 groups

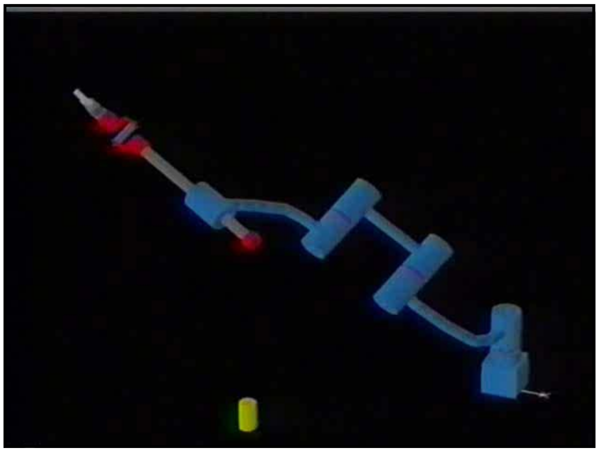
Kinematics, Dynamics & Control

Stanford Robotic Platforms

Romeo & Juliet (1993)

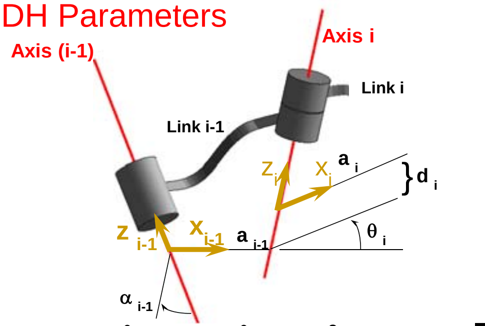
The PUMA Robot





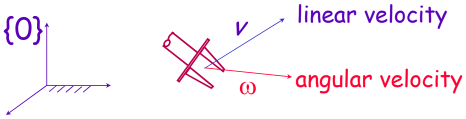
The PUMA Robot Simulator

DH Parameters



$$
{}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Basic Jacobian

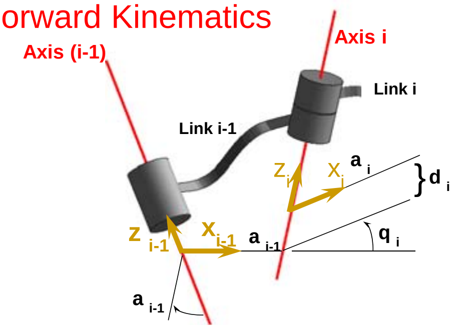


$$
\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J_0(q)_{(6xn)}\,\dot{q}_{(nx1)}
$$

$$
\dot{x}_P = E_P(x_P)v
$$

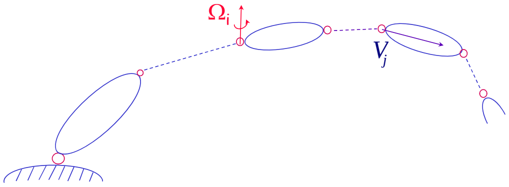
$$
\dot{x}_R = E_R(x_R)\omega
$$

Forward Kinematics



Forward Kinematics: ${}^{0}_{N}T = {}^{0}_{1}T \;\; {}^{1}_{2}T \; \ldots \; {}^{N-1}_{\;\;N}T$

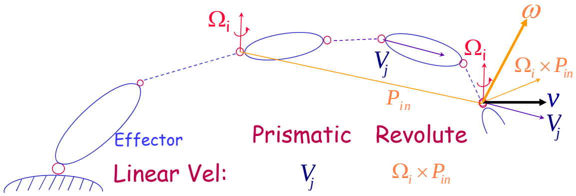
The Jacobian (EXPLICIT FORM)



Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

The Jacobian (EXPLICIT FORM)



	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i\, V_i + \bar{\epsilon}_i\, (\Omega_i \times P_{in})] \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i\, \Omega_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

The Jacobian

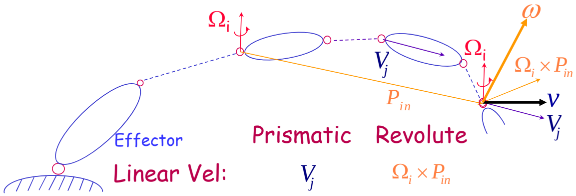
$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_P = \frac{\partial x_P}{\partial q_1}.\dot{q}_1 + \frac{\partial x_P}{\partial q_2}.\dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n}.\dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \end{pmatrix}$$

The Jacobian (EXPLICIT FORM)



	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i\, Z_i + \bar{\epsilon}_i\, (Z_i \times P_{in})] \dot{q}_i \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} (\bar{\epsilon}_i\, Z_i)\, \dot{q}_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

Jacobian in a Frame

Vector Representation

$$J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \\ \bar{\epsilon}_1 . Z_1 & \bar{\epsilon}_2 . Z_2 & \cdots & \bar{\epsilon}_n . Z_n \end{pmatrix}$$

In {0}

$${}^0J = \begin{pmatrix} \frac{\partial\, {}^0x_P}{\partial q_1} & \frac{\partial\, {}^0x_P}{\partial q_2} & \cdots & \frac{\partial\, {}^0x_P}{\partial q_n} \\ \bar{\epsilon}_1 . {}^0Z_1 & \bar{\epsilon}_2 . {}^0Z_2 & \cdots & \bar{\epsilon}_n . {}^0Z_n \end{pmatrix}$$

$$v = [\epsilon_1\, Z_1 + \bar{\epsilon}_1\, (Z_1 \times P_{1n}) \quad \epsilon_2\, Z_2 + \bar{\epsilon}_2\, (Z_2 \times P_{2n}) \quad \cdots] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = [\bar{\epsilon}_1\, Z_1 \quad \bar{\epsilon}_2\, Z_2 \quad \cdots \quad \bar{\epsilon}_n\, Z_n] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

Velocity/Force Duality

$$\boxed{\dot{x} = J\, \dot{\theta}}$$

$$\boxed{\tau = J^T F}$$

Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \left(J_1\ J_2 \cdots J_n\right)$$

$$\det(J) = 0$$

$$\det\left({}^{i}J\right) = \det\left({}^{j}J\right)$$

Jacobian for X

$$\dot{x}_P = J_{X_P}(q)\dot{q} \qquad \begin{pmatrix}\dot{x}_P\\ \dot{x}_R\end{pmatrix} = \begin{pmatrix}J_{X_P}(q)\\ J_{X_R}(q)\end{pmatrix}\dot{q}$$

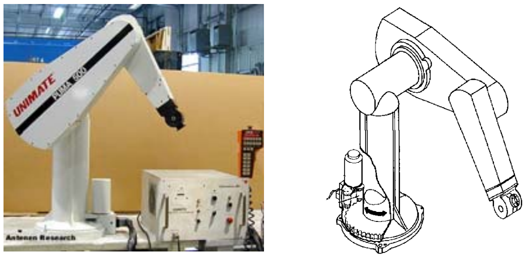
$$\dot{x}_R = J_{X_R}(q)\dot{q}$$

Cartesian & Direction Cosines

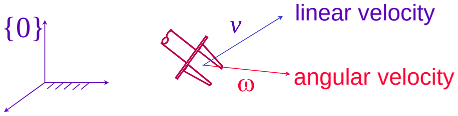
$$\dot{X}_{(12x1)} = J_X(q)_{(12x6)}\,\dot{q}_{(6x1)}$$

The Jacobian is dependent on the representation

The PUMA Robot Singularities



Basic Jacobian



$$\begin{pmatrix}v\\ \omega\end{pmatrix}_{(6x1)} = J_0(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

$$\dot{x}_P = E_P(x_P)\ v$$

$$\dot{x}_R = E_R(x_R)\ \omega$$

Representations

$$x = \begin{bmatrix} x_P\\ x_R\end{bmatrix}$$

- Cartesian
- Spherical
- Cylindrical
- ….

- Euler Angles
- Direction Cosines
- Euler Parameters

Jacobian and Basic Jacobian

$$J = \begin{pmatrix}J_{XP}\\ J_{XR}\end{pmatrix} = \left(\begin{array}{c|c}E_P & 0\\ 0 & E_R\end{array}\right)\begin{pmatrix}J_v\\ J_w\end{pmatrix}$$

$$\qquad\qquad\downarrow\qquad\qquad\downarrow$$

$$\underline{\underline{J(q) \quad = \quad E(x) \quad J_0(q)}}$$

Position Representations

$$\dot{x}_P = E_P(x_P)\ v$$

Cartesian Coordinates (x, y, z)

$$E_P(x) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \quad y \quad z)^T = (\rho\cos\theta \quad \rho\sin\theta \quad z)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ -\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \quad y \quad z)^T = (\rho\cos\theta\sin\phi \quad \rho\sin\theta\sin\phi \quad \rho\cos\theta)^T$$

$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & \_\sin\phi/\rho \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix};\ E_r(x_r) = \begin{pmatrix} -\hat{r}_1 \\ -\hat{r}_2 \\ -\hat{r}_3 \end{pmatrix}$$

$$\dot{x}_r = E_r\omega$$

Position Representations (inverse)

$$v = E_P^{-1}(x)\dot{x}_p$$

Cartesian Coordinates (x, y, z)

$$E_P^{-1}(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ -\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Direction Cosines – Rotation Error

Instantaneous Angular Error

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix};\ x_{rd} = \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

$$\delta x_r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{pmatrix}$$

$$\omega = \frac{1}{2} E^T \dot{x}_r$$

$$\delta\phi = \frac{1}{2} E^T \delta x_r$$

$$\delta x_r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{pmatrix}$$

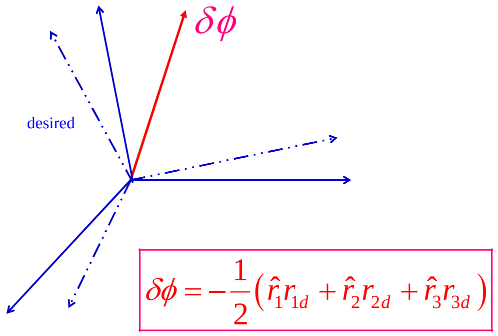
Euler Parameters

$$x_r = \lambda = \left(\lambda_0\, \lambda_1\, \lambda_2\, \lambda_3\right)^T$$

$$\dot{\lambda} = E\omega$$

$$E = \frac{1}{2}\begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

Instantaneous Angular Error



$$\boxed{\delta\phi = -\frac{1}{2}\left(\hat{r}_1 r_{1d} + \hat{r}_2 r_{2d} + \hat{r}_3 r_{3d}\right)}$$

Euler Parameters

$$E_r^+(x_r) = 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

$$\boxed{\delta\phi = E^+ \lambda_d}$$

Euler Angles

$$E_r(X) = \begin{pmatrix} -S\varphi C\theta / S\theta & C\varphi C\theta / S\theta & 1 \\ C\varphi & S\varphi & 0 \\ S\varphi / S\theta & -C\varphi / S\theta & 0 \end{pmatrix}$$

$$E_r^{-1}(x_r) = \begin{pmatrix} 0 & \cos\psi & \sin\psi \sin\theta \\ 0 & \sin\psi & -\cos\psi \sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}$$

Joint Space Dynamics

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma$$

q: Generalized Joint Coordinates

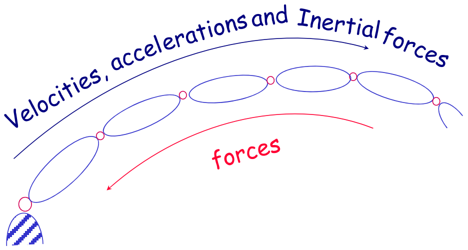
$M(q)$: Mass Matrix - Kinetic Energy Matrix

$V(q,\dot{q})$: Centrifugal and Coriolis forces

$G(q)$: Gravity forces

Γ: Generalized forces

Newton-Euler Algorithm



Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=\tau$$

Lgrangian ↘ $L = K - U$ ← Kinetic Energy; ↖ Potential Energy

Since $U=U(q)$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q}}_{\text{Inertial forces}}+\frac{\partial U}{\partial q}=\tau$$

Gravity vector

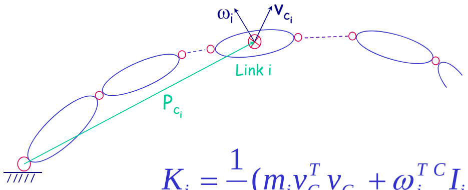
Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q}+\frac{\partial U}{\partial q}=\tau$$

$$\boxed{M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau}$$

$M(q)$: $K=\frac{1}{2}\dot{q}^T M\dot{q}$ $M(q) \Rightarrow V(q,\dot{q})$

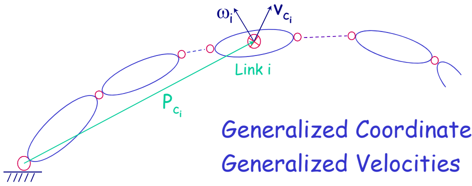
Equations of Motion Explicit Form



$$K_i=\frac{1}{2}(m_i v_{C_i}^T v_{C_i}+\omega_i^T\,{}^C I_i\omega_i)$$

Total Kinetic Energy ⇨ $K=\sum_{i=1}^{n} K_i$

Equations of Motion Explicit Form



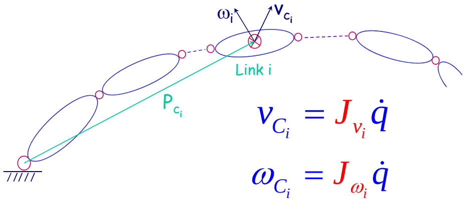
Generalized Coordinates q

Generalized Velocities $\dot{q}$

Kinetic Energy
Quadratic Form of
Generalized Velocities $K=\frac{1}{2}\dot{q}^T M\dot{q}$

$$\frac{1}{2}\dot{q}^T M\,\dot{q}\equiv\frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i}+\omega_i^T\,{}^C I_i\omega_i)$$

Equations of Motion Explicit Form



$$v_{C_i}=J_{v_i}\dot{q}$$

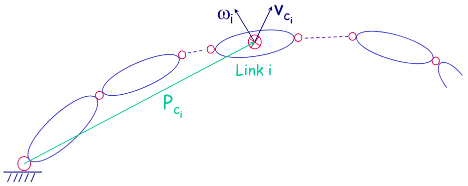
$$\omega_{C_i}=J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T M\,\dot{q}=\frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i}+\omega_i^T\,{}^C I_i\omega_i)$$

$$=\frac{1}{2}\sum_{i=1}^{n}(m_i\dot{q}^T J_{v_i}^T J_{v_i}\dot{q}+\dot{q}^T J_{\omega_i}^T\,{}^C I_i J_{\omega_i}\dot{q})$$

Equations of Motion

Explicit Form



$$\frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\, {}^C I_i J_{\omega_i})\right] \dot{q}$$

$$\boxed{M = \sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\, {}^C I_i J_{\omega_i})}$$

Robot Control

Natural Systems

Conservative Forces

$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial \dot{x}}\right) - \frac{\partial(K-V)}{\partial x} = 0$$

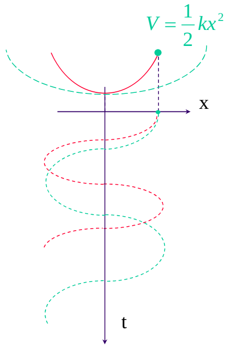
$$K = \frac{1}{2} m \dot{x}^2$$

$$m\ddot{x} + kx = 0$$

$$V = \frac{1}{2} k x^2$$

Natural Systems

Conservative Forces



$$m\ddot{x} + kx = 0$$

Frequency increases
with stiffness
and inverse mass

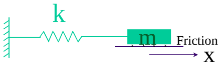
Natural Frequency $\omega_n = \sqrt{\frac{k}{m}}$

$$\ddot{x} + \omega_n^2 x = 0$$

$$x(t) = c\ \cos(\omega_n t + \phi)$$

Natural Systems

Dissipative Systems

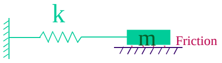


$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial \dot{x}}\right) - \frac{\partial(K-V)}{\partial x} = f_{friction}$$

Viscous friction: $f_{friction} = -b\dot{x}$

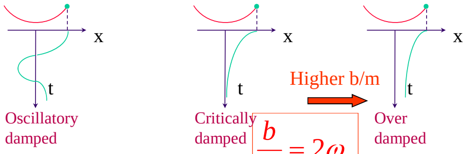
$$m\ddot{x} + b\dot{x} + kx = 0$$

Dissipative Systems



$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$



$$\boxed{\frac{b}{m} = 2\omega_n}$$

$2^{\underline{d}}$ order systems

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\frac{b/m}{2\omega_n}\cdot 2\omega_n$$

$$\omega_n^{\ 2}$$

Natural damping ratio

Critically damped when b/m=2w$_n$

$$\xi_n = \frac{b}{2\omega_n m} = \frac{b}{2\sqrt{km}}$$

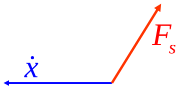
Critically damped system: $\xi_n = 1 \ \ (b = 2\sqrt{km})$

Asymptotic Stability

a system $\dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot{x}}\right) - \dfrac{\partial\left(K - V_{goal}\right)}{\partial x} = 0\ F_s$

is asymptotically stable if

$$\boxed{F_s^{\ T}\dot{x} < 0 \quad ; \ for \ \dot{x} \neq 0}$$



$$F_s = -k_v\dot{x} \ \rightarrow \ k_v > 0$$

Control

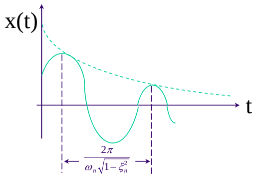
$$F = -k_p\left(x - x_{goal}\right) - k_v\dot{x}$$

Time Response

$$\ddot{x} + 2\xi_n\omega_n\dot{x} + \omega_n^2 x = 0$$

Natural frequency $\omega_n = \sqrt{\dfrac{k}{m}}$; $\xi_n = \dfrac{b}{2\sqrt{km}}$ Natural damping ratio

$$x(t) = ce^{-\xi_n\omega_n t}\cos(\underbrace{\omega_n\sqrt{1-\xi_n^2}}_{\omega}t + \phi)$$



damped Natural frequency

$$\omega = \omega_n\sqrt{1-\xi_n^2}$$

Proportional-Derivative Control (PD)

$$m\ddot{x} = f = -k_p(x - x_d) - k_v\dot{x}$$

$$m\ddot{x} + k_v\dot{x} + k_p(x - x_d) = 0$$

Velocity gain Position gain

$$1.\ \ddot{x} + \frac{k_v}{m}\dot{x} + \frac{k_p}{m}(x - x_d) = 0$$

$$1.\ \ddot{x} + 2\xi\omega\dot{x} + \omega^2(x - x_d) = 0$$

$\xi = \dfrac{k_v}{2\sqrt{k_p m}}$ closed loop damping ratio

$\omega = \sqrt{\dfrac{k_p}{m}}$ closed loop frequency

Passive Systems (Stability)

$$V_{goal} = \frac{1}{2}k_p\left(x - x_g\right)^T\left(x - x_g\right)$$

System $\dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot{x}}\right) - \dfrac{\partial K}{\partial x} = f$

$$\Downarrow \quad f = -\frac{\partial V_{goal}}{\partial X}$$

Conservative Forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial\left(K - V_{goal}\right)}{\partial x} = 0$$

Stable

Gains

$$k_p = m\omega^2$$

$$k_v = m(2\xi\omega)$$

Gain Selection

$$\text{set} \begin{pmatrix}\xi \\ \omega\end{pmatrix} \rightarrow \begin{matrix} k_p = m\omega^2 \\ k_v = m(2\xi\omega)\end{matrix}$$

Unit mass system

$$k'_p = \omega^2$$

$$k'_v = 2\xi\omega$$

m - mass system

$$k_p = m\ k'_p$$

$$k_v = m\ k'_v$$

Control Partitioning

$$m\ddot{x} = f \Longrightarrow m\ (1.\ddot{x}) = m\ \ f'$$

$$f = -k_v\dot{x} - k_p(x - x_d)$$

$$f = m[\underbrace{-k'_v\dot{x} - k'_p(x - x_d)}_{f'}] = m\,f'$$

$$m\ddot{x} = m\,f'$$

$1.\,\ddot{x} = f'$ unit mass system

$$1.\ddot{x} + \underset{\overset{\Uparrow}{2\xi\omega}}{k'_v}\dot{x} + \underset{\overset{\Uparrow}{\omega^2}}{k'_p}(x - x_d) = 0$$

Non Linearities

$$m\ddot{x} + b(x,\dot{x}) = f$$

Control Partitioning

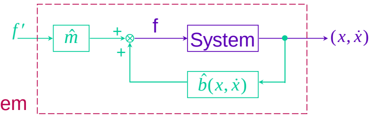
$$f = \alpha f' + \beta$$

with $\alpha = \hat{m}$

$$\beta = \hat{b}(x,\dot{x})$$

$$m\ddot{x} + b(x,\dot{x}) = \hat{m}f' + \hat{b}(x,\dot{x})$$

$$\longrightarrow 1.\,\ddot{x} = f'$$



Unit mass system

Motion Control

$$m\ddot{x} + b(x,\dot{x}) = \underset{f\,=\,mf'+b}{f} \Rightarrow 1.\,\ddot{x} = f'$$

Goal Position (x_d):

Control: $f' = -k'_v\dot{x} - k'_p(x - x_d)$

Closed-loop System: $1.\,\ddot{x} + k'_v\dot{x} + k'_p(x - x_d) = 0$

Trajectory Tracking

$$x_d(t);\ \dot{x}_d(t);\ \text{and } \ddot{x}_d(t)$$

Control: $f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d)$

Closed-loop System:

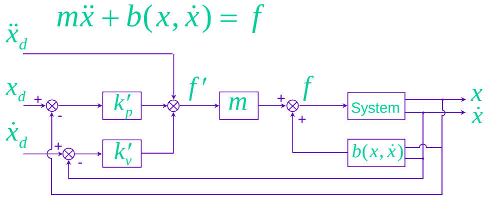
$$(\ddot{x} - \ddot{x}_d) + k'_v(\dot{x} - \dot{x}_d) + k'_p(x - x_d) = 0$$

with $e \equiv x - x_d$

$$\ddot{e} + k'_v\dot{e} + k'_p e = 0$$

Disturbance Rejection

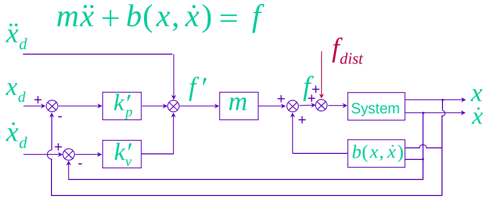
$$m\ddot{x} + b(x,\dot{x}) = f$$



$$\ddot{e} + k'_v\dot{e} + k'_p e = 0$$

Disturbance Rejection

$$m\ddot{x} + b(x,\dot{x}) = f$$



$$m\ddot{x} + b(x,\dot{x}) = f + f_{dist} \longleftarrow$$

bounded

$$\{\forall t \left| f_{dist} \right| < a\}$$

Control $f = mf' + b(x,\dot{x})$

Closed loop

$$\boxed{\ddot{e} + k'_v\dot{e} + k'_p e = \frac{f_{dist}}{m}}$$

Steady-State Error

$$\ddot{e} + k'_v\dot{e} + k'_p e = \frac{f_{dist}}{m}$$

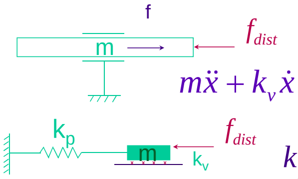
The steady-state $\left(\dot{e} = \ddot{e} = 0\right)$:

$$k'_p e = \frac{f_{dist}}{m}$$

$$\underline{\underline{e = \frac{f_{dist}}{mk'_p} = \frac{f_{dist}}{k_p}}} \longleftarrow$$

Closed loop
position
gain (stiffness)

Steady-State Error - Example



$$m\ddot{x} + k_v\dot{x} + k_p(x - x_d) = 0$$

$$k_p(x - x_d) = f_{dist}$$

$$x = x_d + \underbrace{\frac{f_{dist}}{k_p}}_{\Delta x}$$

$$f_{dist} = k_p \Delta x$$

$$\Delta x = \frac{f_{dist}}{k_p}$$

Closed Loop Stiffness

Effective Inertia

$$I_{eff} = I_L + \eta^2 I_m$$

for a manipulator
$I_L = I_L(q)$

$\eta = 1$
Direct Drive

Gain Selection

$$k_p = (I_L + \eta^2 I_m)k'_p$$

$$k_v = (I_L + \eta^2 I_m)k'_v$$

Time Optimal Selection

$$\hat{I}_L = \frac{1}{4}(\sqrt{I_{L_{\min}}} + \sqrt{I_{L_{\max}}})^2$$

PID (adding Integral action)

System $\quad m\ddot{x} + b(x,\dot{x}) = f + f_{dist}$

Control $\quad f = mf' + b(x,\dot{x})$

$$f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d) - k'_i\int(x - x_d)dt$$

Closed-loop System

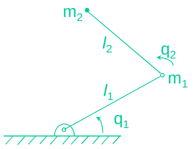
$$\ddot{e} + k'_v\dot{e} + k'_p e + k'_i\int e\,dt = \frac{f_{dist}}{m}$$

$$\dddot{e} + k'_v\ddot{e} + k'_p\dot{e} + k'_i e = 0$$

constant

Steady-state Error $\boxed{e = 0}$

Manipulator Control



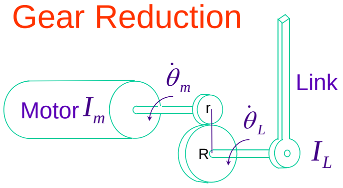
$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau$$

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} m_{112} \\ 0 \end{pmatrix}\begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 & m_{122} \\ -\frac{m_{112}}{2} & 0 \end{pmatrix}\begin{pmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

$$\underline{m_{11}\ddot{\theta}_1} + m_{12}\ddot{\theta}_2 + m_{112}\dot{\theta}_1\dot{\theta}_2 + m_{122}\dot{\theta}_2^2 + G_1 = \underline{\tau_1}$$

$$\underline{m_{22}\ddot{\theta}_2} + m_{21}\ddot{\theta}_1 - \frac{m_{112}}{2}\dot{\theta}_1^2 + G_2 = \underline{\tau_2}$$

Gear Reduction



Gear ratio $\eta = \frac{R}{r}$

$$\dot{\theta}_L = (\frac{1}{\eta})\dot{\theta}_m$$

$$\tau_L = \eta\tau_m$$

$$\tau_m = I_m\ddot{\theta}_m + \frac{1}{\eta}(I_L\ddot{\theta}_L) + b_m\dot{\theta}_m + \frac{1}{\eta}b_L\dot{\theta}_L$$

$$\ddot{\theta}_L = \frac{1}{\eta}\ddot{\theta}_m$$

$$\tau_m = (I_m + \frac{I_L}{\eta^2})\ddot{\theta}_m + (b_m + \frac{b_L}{\eta^2})\dot{\theta}_m$$

$$\tau_L = (\underbrace{I_L + \eta^2 I_m}_{\text{Effective Inertia}})\ddot{\theta}_L + (\underbrace{b_L + \eta^2 b_m}_{\text{Effective Damping}})\dot{\theta}_L$$

KUKA youBot

PD Control Stability

$$\underbrace{M(q)\ddot{q}+B(q)[\dot{q}\dot{q}]+C(q)[\dot{q}^2]}+G(\theta)=\tau$$

$$\Downarrow \qquad \tau=-k_p(q-q_d)-k_v\dot{q}$$

$$V_d=1/2k_p(q-q_d)^2 \ \Downarrow$$

$$\overbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q}}+\frac{\partial V_s}{\partial q}=\cancel{\tau}\frac{\partial V_d}{\partial q}-k_v\dot{q}$$

Nonlinear Dynamic Decoupling

$$M(\theta)\ddot{\theta}+V(\theta,\dot{\theta})+G(\theta)=\tau$$

$$\tau=\hat{M}(\theta)\underline{\tau'}+\hat{V}(\theta,\dot{\theta})+\hat{G}(\theta)$$

$$\mathbf{1}.\ddot{\theta}=(M^{-1}\hat{M})\tau'+M^{-1}[(V-\hat{V})+(G-\hat{G})]$$

with perfect estimates

$$\mathbf{1}.\ddot{\theta}=\tau'+\varepsilon(t)$$

τ' : input of the unit-mass systems

$$\tau'=\ddot{\theta}_d-k_v'(\dot{\theta}-\dot{\theta}_d)-k_p'(\theta-\theta_d)$$

Closed-loop

$$\ddot{E}+k_v'\dot{E}+k_p'E=0+\varepsilon(t)$$

PD Control Stability

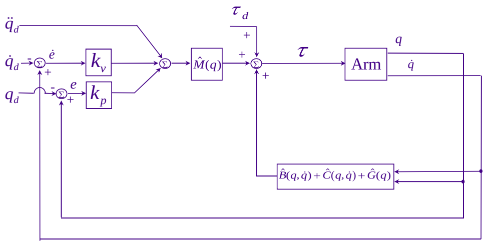
$$\underbrace{M(q)\ddot{q}+B(q)[\dot{q}\dot{q}]+C(q)[\dot{q}^2]}+G(\theta)=\tau$$

$$\Downarrow \qquad \tau=-k_p(q-q_d)-k_v\dot{q}$$

$$V_d=1/2k_p(q-q_d)^T(q-q_d) \ \Downarrow$$

$$\overbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q}}+\frac{\partial (V_s-V_d)}{\partial q}=\tau_s$$

$\tau_s=-k_v\dot{q}$ with $\boxed{\tau_s^T\dot{q}<0 \ \ for\ \dot{q}\neq 0; k_v>0}$



Performance

High Gains ⟶ better disturbance rejection

Gains are limited by

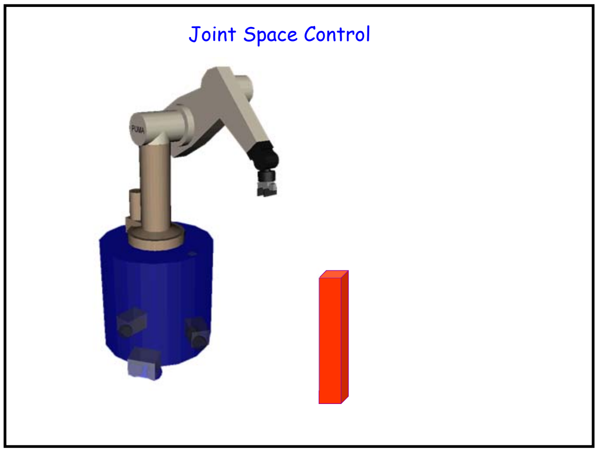
structural flexibilities
time delays (actuator-sensing)
sampling rate

$$\omega_n \le \frac{\omega_{res}}{2} \longleftarrow \text{lowest structural flexibility}$$

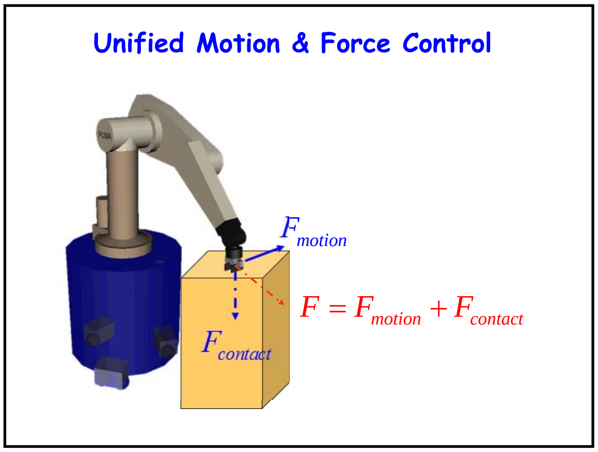
$$\omega_n \le \frac{\omega_{delay}}{3} \longleftarrow \text{largest delay} \left(\frac{2\pi}{\tau_{delay}}\right)$$

$$\omega_n \le \frac{\omega_{sampling-rate}}{5}$$

Task Oriented Control

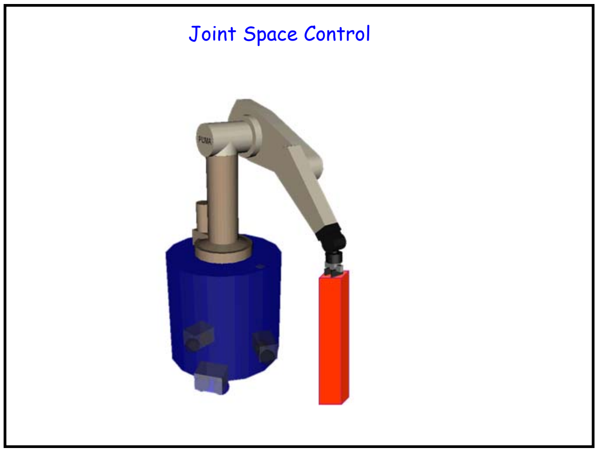


Joint Space Control

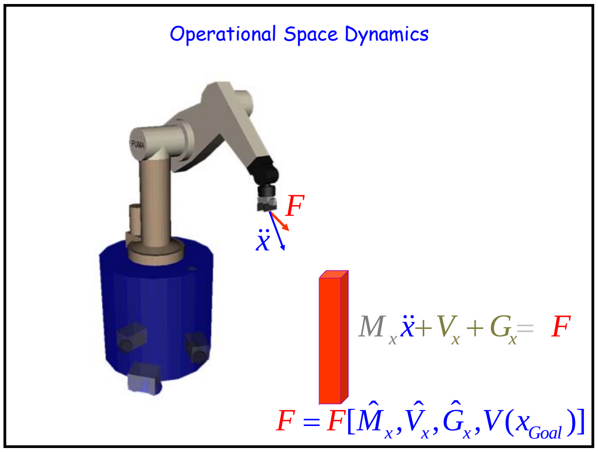


Unified Motion & Force Control

$F = F_{motion} + F_{contact}$



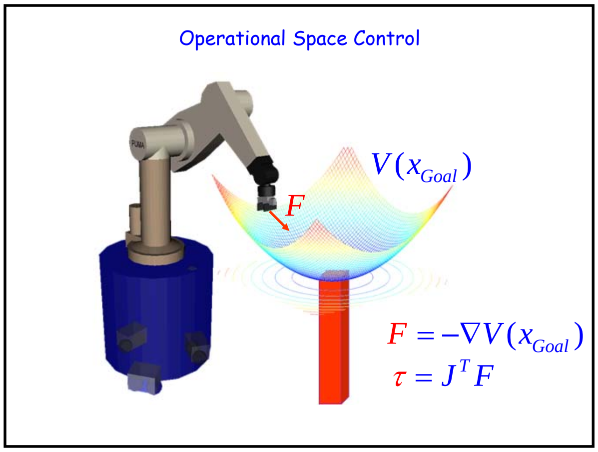
Joint Space Control



Operational Space Dynamics

$M_x\ddot{x} + V_x + G_x = F$

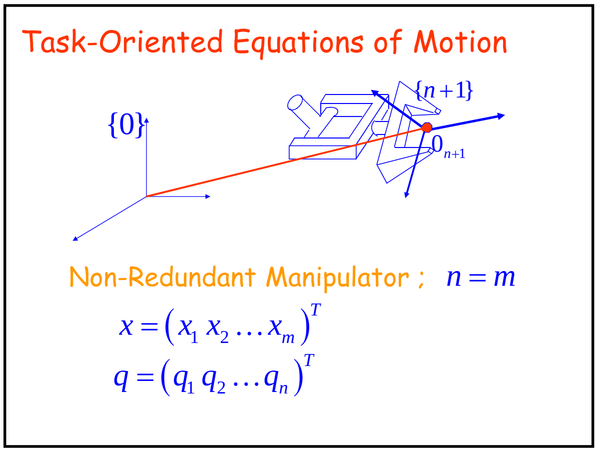
$F = F[\hat{M}_x, \hat{V}_x, \hat{G}_x, V(x_{Goal})]$



Operational Space Control

$F = -\nabla V(x_{Goal})$

$\tau = J^T F$



Task-Oriented Equations of Motion

Non-Redundant Manipulator ; $n = m$

$x = (x_1\, x_2 \ldots x_m)^T$

$q = (q_1\, q_2 \ldots q_n)^T$

Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F \qquad x = \begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

with $L(x, \dot{x}) = K(x, \dot{x}) - U(x)$

Joint Space/Task Space Relationships

$$M_x(x) = J^{-T}(q)\, M(q)\, J^{-1}(q)$$

$$V_x(x, \dot{x}) = J^{-T}(q) V(q, \dot{q}) - M_x(q)\, h(q, \dot{q})$$

$$G_x(x) = J^{-T}(q)\, G(q)$$

where $h(q, \dot{q}) \doteq \dot{J}(q)\dot{q}$

Operational Space Dynamics

$$M_x(x)\ddot{x} + V_x(x, \dot{x}) + G_x(x) = F$$

$x:$ End-Effector Position and Orientation

$M_x(x):$ End-Effector Kinetic Energy Matrix

$V_x(x, \dot{x}):$ End-Effector Centrifugal and Coriolis forces

$G_x(x):$ End-Effector Gravity forces

$F:$ End-Effector Generalized forces

Joint Space/Task Space Relationships

Kinetic Energy

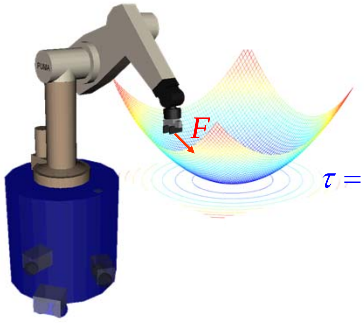
$$K_x(x, \dot{x}) \equiv K_q(q, \dot{q})$$

$$\frac{1}{2}\dot{x}^T M_x(x)\dot{x} \equiv \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

Using $\dot{x} = J(q)\dot{q}$

$$\boxed{\frac{1}{2}\dot{q}^T (J^T M_x J)\, \dot{q} \equiv \frac{1}{2}\dot{q}^T M\, \dot{q}}$$

End-Effector Control



$\tau = J^T(q)F$

Passive Systems (Stability)

$$V_{goal} = \frac{1}{2} k_p \left(x - x_g \right)^T \left(x - x_g \right)$$

System $$\frac{d}{dt}\left(\frac{\partial (K-V)}{\partial \dot{x}} \right) - \frac{\partial (K-V)}{\partial x} = F$$

$$\Downarrow \; F = -\frac{\partial}{\partial X}\left(V_{goal} - \hat{V} \right)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial \left(K - V_{goal} \right)}{\partial x} = 0$$

Conservative Forces

$\boxed{\text{Stable}}$

Perfect Estimates

$$I\,\ddot{x} = F'$$

F' input of decoupled end-effector

Goal Position Control

$$F' = -k_v'\,\dot{x} - k_p' \left(x - x_g \right)$$

Closed Loop

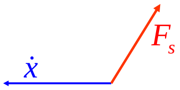
$$I\,\ddot{x} + k_v'\dot{x} + k_p'(x - x_g) = 0$$

Asymptotic Stability

a system $$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial \left(K - V_{goal} \right)}{\partial x} = 0 \; F_s$$

is asymptotically stable if

$$\boxed{F_s^T \dot{x} \;<\; 0 \quad ; \; for \;\; \dot{x} \neq 0}$$



$$F_s = -k_v \dot{x} \;\; \rightarrow \;\; k_v > 0$$

Control

$$F = -k_p \left(x - x_{goal} \right) + \hat{G}_x - k_v \dot{x}$$

Trajectory Tracking

Trajectory: x_d , $\dot{x}_d$, $\ddot{x}_d$

$$F' = I\,\ddot{x}_d - k_v'(\dot{x} - \dot{x}_d) - k_p'(x - x_d)$$

$$(\ddot{x} - \ddot{x}_d) + k_v'(\dot{x} - \dot{x}_d) + k_p'(x - x_d) = 0$$

or $$\boxed{\ddot{\varepsilon}_x + k_v'\dot{\varepsilon}_x + k_p'\varepsilon_x = 0}$$

with $\varepsilon_x = x - x_d$

Nonlinear Dynamic Decoupling

Model

$$M_x(x)\ddot{x} + V_x(x,\dot{x}) + G_x(x) \;=\; F$$

Control Structure

$$F = \hat{M}(x)F' + \hat{V}_x(x,\dot{x}) + \hat{G}_x(x)$$

Decoupled System

$$I\,\ddot{x} = F'$$

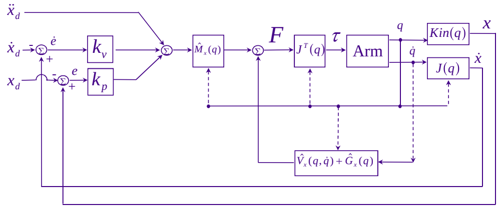
with $\tau = J^T F$

In joint space

$$\ddot{\varepsilon}_q + k_v'\dot{\varepsilon}_q + k_p'\varepsilon_q = 0$$

with $\varepsilon_q = q - q_d$

Task-Oriented Control



Stiffness

$$\ddot{z} + k_v'\dot{z} + k_{p_z}'(z - z_d) = 0$$

determines stiffness along z

Closed-Loop Stiffness: $\hat{M}_x k_p' = k_p$

$$F = K_x(x - x_d)$$

$$\tau = J^T F = J^T K_x \Delta x = (J^T K_x J)\Delta\theta = K_\theta \Delta\theta$$

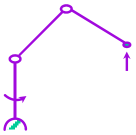
$$\boxed{K_\theta = J^T(\theta) K_x J(\theta)}$$

Compliance

$$\boxed{I\ \ddot{x} = F'}$$

$$F' = -\begin{pmatrix} k'_{p_x} & 0 & 0 \\ 0 & k'_{p_y} & 0 \\ 0 & 0 & k'_{p_z} \end{pmatrix}(x - x_d) - k'_v \dot{x}$$

set to zero



$$\ddot{x} + k_v'\dot{x} + k_{px}'(x - x_d) = 0$$

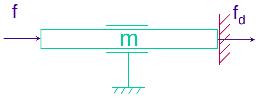
$$\ddot{y} + k_v'\dot{y} + k_{py}'(y - y_d) = 0$$

$$\ddot{z} + k_v'\dot{z} = 0$$

Compliance along Z

Force Control

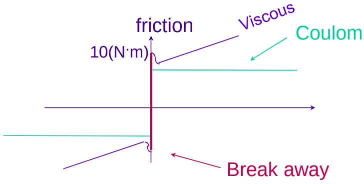
1-d.o.f.



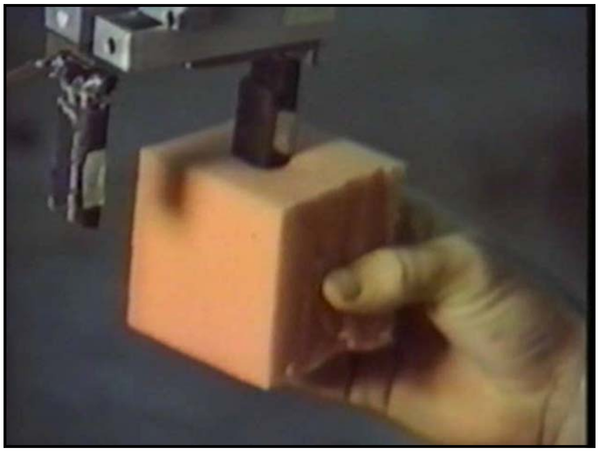
$$m\ddot{x} = f$$

set $f = f_d$

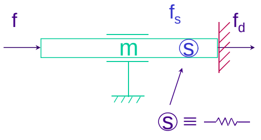
Problem



$f_d = 1Nm$ → output≃0



Force Sensing



$; f_s = k_s x$

$$m\ddot{x} + \underbrace{k_s x}_{f_s} = f$$

At static Equilibrium

$$f_s = f_d \Rightarrow f = f_d$$

Dynamics

$$m\ddot{x} + k_s x = f_d + f_{Dynamic}$$

Dynamics

$$m\ddot{x} + \underbrace{k_s x}_{f_s} = f \qquad f_s = k_s x$$

$$\dot{f}_s = k_s \dot{x}$$

$$\ddot{f}_s = k_s \ddot{x}$$

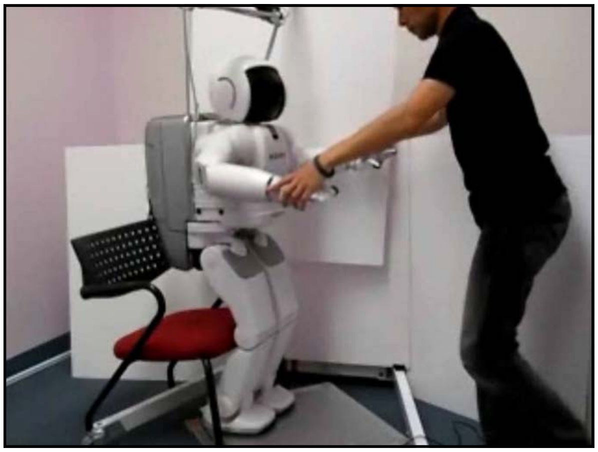
$$\frac{m}{k_s}\ddot{f}_s + f_s = f$$

Control

$$f_d + \frac{m}{k_s}(-k'_{p_f}(f_s - f_d) - k'_{v_f}\dot{f}_s)$$

Closed Loop

$$\frac{m}{k_s}[\ddot{f}_s + k'_{v_f}\dot{f}_s + k'_{p_f}(f_s - f_d)] + f_s = f_d$$



Steady-State error

$$\frac{m}{k_s}(\ddot{f}_s + k'_{v_f}\dot{f}_s + k'_{p_f}(f_s - f_d)) + (f_s - f_d) = 0$$

$$\uparrow$$
$$f_{dist}$$

$$\ddot{f}_s = \dot{f}_s = 0$$

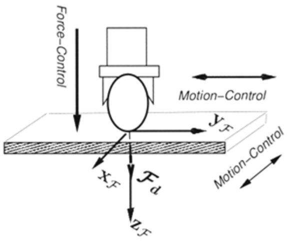
$$(\frac{mk'_{p_f}}{k_s} + 1)e_f = f_{dist}$$

$$e_f = \frac{f_{dist}}{1 + \frac{mk'_{p_f}}{k_s}}$$

Task Specification

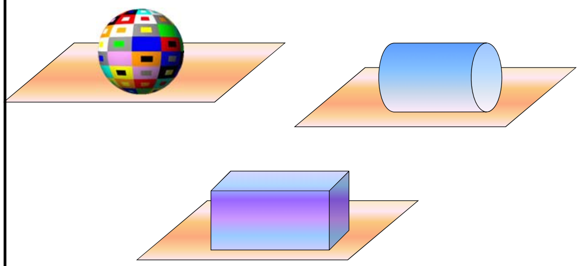
$$F = \Omega F_{motion} + \bar{\Omega} F_{force}$$

Selection matrix

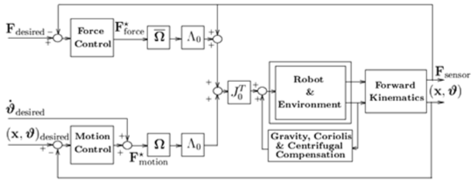


$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \bar{\Omega} = I - \Omega$$

Task Description



Unified Motion & Force Control



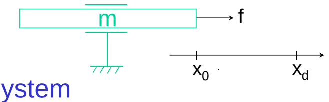
Two decoupled Subsystems

$$\Omega\dot{\vartheta} = \Omega F^*_{motion}$$

$$\bar{\Omega}\dot{\vartheta} = \bar{\Omega} F^*_{force}$$

System Identification

Identification



System

$$m\ddot{x} + b\dot{x} = f$$

Control

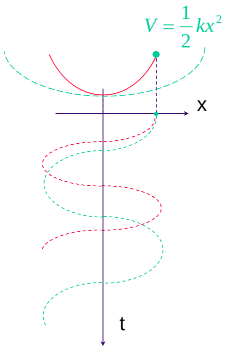
$$f = -k_p(x - x_d) - k_v\dot{x}$$

Closed-Loop

$$m\ddot{x} + (b + k_v)\dot{x} + k_p(x - x_d) = 0$$

Natural Systems

Conservative Systems



$$m\ddot{x} + kx = 0$$

Frequency increases
with stiffness
and inverse mass

Natural Frequency $\omega_n = \sqrt{\frac{k}{m}}$

$$\ddot{x} + \omega_n^2 x = 0$$

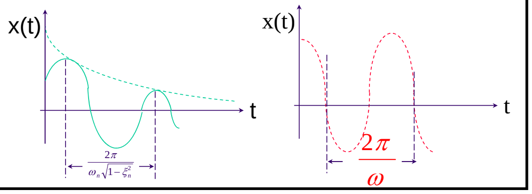
$$x(t) = c\ \cos(\omega_n t + \phi)$$

Time Response

$$\ddot{x} + 2\xi_n\omega_n\dot{x} + \omega_n^2 x = 0$$

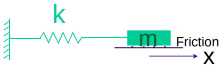
$$\omega_n = \sqrt{\frac{k_p}{m}} \Rightarrow m = \frac{k_p}{\omega_n^2} \qquad \xi_n = \frac{b + k_v}{2\sqrt{k_p m}}$$

$$x(t) = ce^{-\xi_n\omega_n t}\cos(\omega_n\sqrt{1-\xi_n^2}\,t + \phi)$$



Natural Systems

Dissipative Systems



$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial\dot{x}}\right) - \frac{\partial(K-V)}{\partial x} = f_{friction}$$

Viscous friction: $f_{friction} = -b\dot{x}$

$$m\ddot{x} + b\dot{x} + kx = 0$$