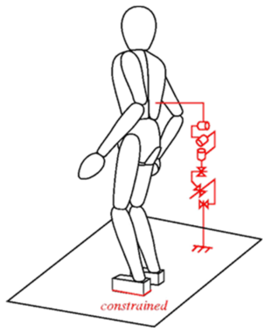
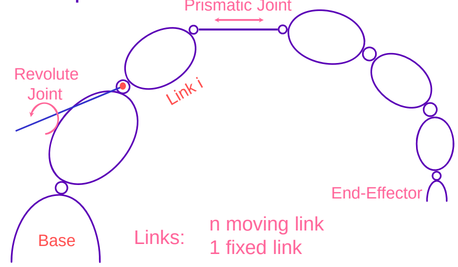
Kinematics



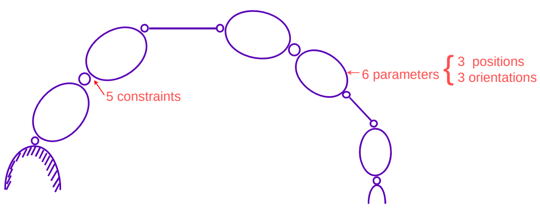
Manipulator



Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

Generalized Coordinates



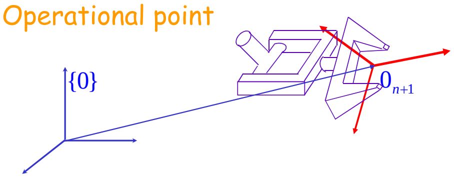
n moving links: 6n parameters

n 1 d.o.f. joints: 5n constraints

d.o.f. (system): 6n - 5n = n

Operational Coordinates

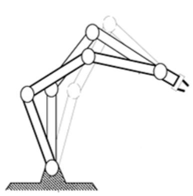
O_{n+1} : Operational point



A set $x_1, x_2, \ldots, x_{m_0}$
of m_0 independent configuration parameters

m_0 : number of degrees of freedom of the end-effector.

Redundancy



A robot is said to be redundant if

$n > m_0$

Degrees of redundancy: $n - m_0$

Task Redundancy

$n > m_{task}$

$n - m_{task}$: degrees of redundancy/task

Geometric Model

Homogeneous Transformation

→ Compact representation

$$\underline{x} = R\underline{x}' + \rho$$

$$\begin{pmatrix} \underline{x} \\ 1 \end{pmatrix} = \begin{pmatrix} R & \rho \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{x}' \\ 1 \end{pmatrix}$$

$$\underline{x} = T\,\underline{x}' \quad ; \quad T_{4\times 4} = \begin{pmatrix} R & \rho \\ 0 & 1 \end{pmatrix}$$

→ T: not orthonormal

$$T^{-1} = \begin{pmatrix} R^T & -R^T\rho \\ 0 & 1 \end{pmatrix}$$

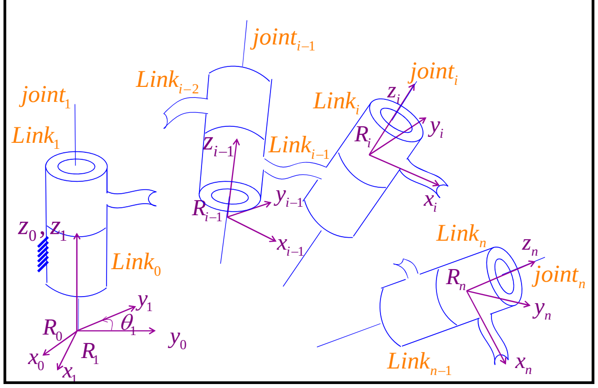
$$\underline{x} = T^{-1}\,\underline{x}'$$

Consecutive Transformations

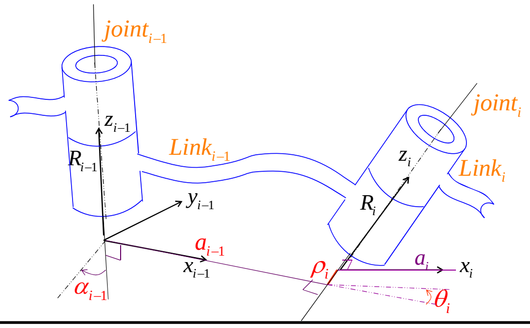
$$T_{1k} = T_1\; T_2\; \dots\; T_k$$

- Forward Kinematics
- Inverse Kinematics

Kinematic Chain



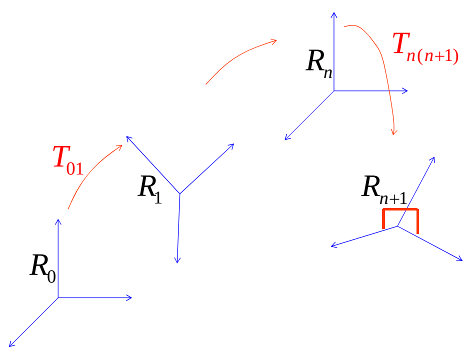
Denavit-Hartenberg (DH) Parameters



Homogeneous Transformation

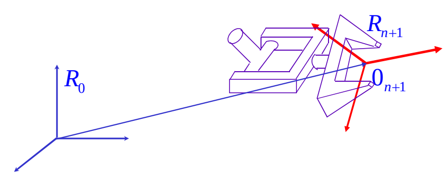
$$T_{(i-1)i} = T_{(i-1)i}\left(\alpha_{i-1}, a_{i-1}, \theta_i, \rho_i\right)$$

$$T_{(i-1)i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{(i-1)} \\ \sin\theta_i\cos\alpha_{(i-1)} & \cos\theta_i\cos\alpha_{(i-1)} & -\sin\alpha_{(i-1)} & -\rho_i\sin\alpha_{(i-1)} \\ \sin\theta_i\sin\alpha_{(i-1)} & \cos\theta_i\sin\alpha_{(i-1)} & \cos\alpha_{(i-1)} & \rho_i\cos\alpha_{(i-1)} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Geometric Model
Forward Kinematics

$$T_{0(n+1)}(q) = T_{01}(q_1)T_{12}(q_2)\ldots T_{(n-1)n}(q_n)T_{n(n+1)}$$



$$T_{0(n+1)}(q) = \begin{pmatrix} R_{o(n+1)}(q) & \rho_{o(n+1)}(q) \\ 0 & 1 \end{pmatrix}$$

m equations

$$x = G(q)$$

$$x = \begin{pmatrix} x_p(q) \\ x_r(q) \end{pmatrix}$$

Representations

$$x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

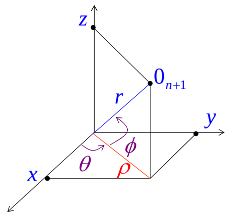
x_P:
- Cartesian
- Spherical
- Cylindrical
- ….

x_R:
- Euler Angles
- Direction Cosines
- Euler Parameters

Position Representations

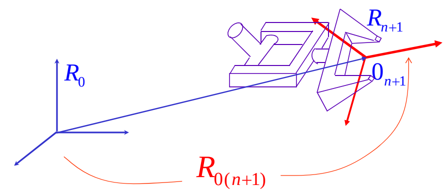
$x_p(q)$ is obtained from $\rho_{0(n+1)}(q)$

- Cartesian (x, y, z)
- Cylindrical (ρ, θ, z)
- Spherical (r, θ, ϕ)



Orientation Representation

$x_R(q)$ is obtained from $R_{0(n+1)}(q)$



$$R_{0(n+1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Direction Cosines

$$R_{0(n+1)} = \begin{pmatrix} r_1(q) & r_2(q) & r_3(q) \end{pmatrix};$$

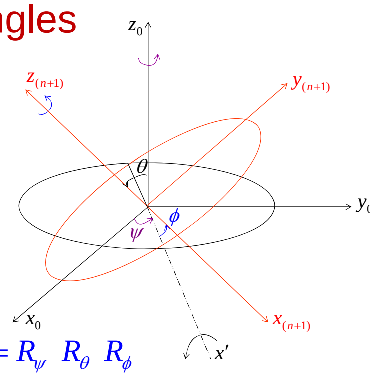
$$x_r = \begin{pmatrix} r_1(q) \\ r_2(q) \\ r_3(q) \end{pmatrix}$$

Euler Angles

$\psi / z_0 \rightarrow x'$

$\theta / x' \rightarrow z_{(n+1)}$

$\phi / z_{(n+1)}$



$$R_{0(n+1)} = R_\psi \; R_\theta \; R_\phi$$

$$R_{0(n+1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$R_{0(n+1)}(q) = \begin{pmatrix} c\psi c\phi - s\psi c\theta s\phi & -c\psi s\phi - s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi - c\psi c\theta s\phi & -s\psi s\phi + c\psi c\theta c\phi & -c\psi s\theta \\ s\theta s\phi & s\theta c\phi & c\theta \end{pmatrix}$$

$$x_r = \begin{pmatrix} \psi(q) \\ \theta(q) \\ \phi(q) \end{pmatrix}$$

$$\psi(q) = \text{sgn}(r_{13}) \arccos\left(-r_{23} \Big/ \sqrt{1 - r_{33}^2}\right);$$

$$\theta(q) = \arccos(r_{33});$$

$$\phi(q) = \text{sgn}(r_{31}) \arccos\left(-r_{32} \Big/ \sqrt{1 - r_{33}^2}\right);$$

$r_{33} \neq \pm 1$

Singularity of the representation

$\theta = k\pi$ k:(integer) $\Rightarrow$ $r_{33} \neq \pm 1$

$(\psi + \phi)$ or $(\psi - \phi)$ are defined-

Euler Parameters

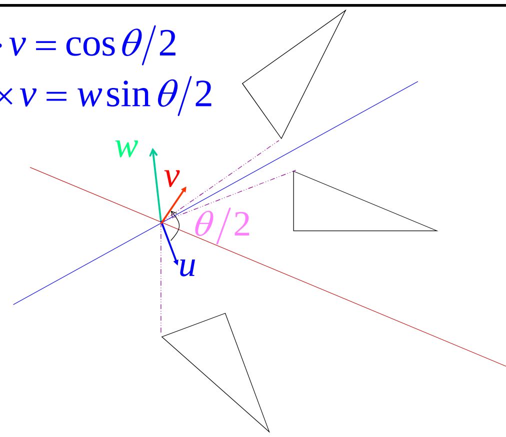
Quaternion

Rotations:

Product of two plane symmetries

$$u \cdot v = \cos\theta/2$$

$$u \times v = w \sin\theta/2$$



Euler 4-Parameters

$$\lambda_0 = \cos\theta/2\ ;$$
$$\lambda_1 = w_1 \sin\theta/2\ ;$$
$$\lambda_2 = w_2 \sin\theta/2\ ;$$
$$\lambda_3 = w_3 \sin\theta/2\ ;$$

Normality condition

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

Rotation Matrix

$$R_{0(n+1)}(q) = \begin{pmatrix} 2\left(\lambda_0^2+\lambda_1^2\right)-1 & 2\left(\lambda_1\lambda_2-\lambda_0\lambda_3\right) & 2\left(\lambda_1\lambda_3+\lambda_0\lambda_2\right) \\ 2\left(\lambda_1\lambda_2+\lambda_0\lambda_3\right) & 2\left(\lambda_0^2+\lambda_2^2\right)-1 & 2\left(\lambda_2\lambda_3-\lambda_0\lambda_1\right) \\ 2\left(\lambda_1\lambda_3-\lambda_0\lambda_2\right) & 2\left(\lambda_2\lambda_3+\lambda_0\lambda_1\right) & 2\left(\lambda_0^2+\lambda_3^2\right)-1 \end{pmatrix}$$

$$R_{0(n+1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Sign Determination:

$(\lambda_0\lambda_1), (\lambda_0\lambda_2), (\lambda_0\lambda_3), (\lambda_1\lambda_2), (\lambda_1\lambda_3)$ and $(\lambda_2\lambda_3)$

$$\lambda_3 = \frac{\eta}{2}\sqrt{r_{11}+r_{22}+r_{33}+1}\ ;$$
$$\lambda_2 = \frac{\eta}{2}\operatorname{sgn}\left(r_{21}-r_{12}\right)\sqrt{-r_{11}-r_{22}+r_{33}+1}\ ;$$
$$\lambda_1 = \frac{\eta}{2}\operatorname{sgn}\left(r_{13}-r_{31}\right)\sqrt{-r_{11}+r_{22}-r_{33}+1}\ ;$$
$$\lambda_0 = \frac{\eta}{2}\operatorname{sgn}\left(r_{32}-r_{23}\right)\sqrt{r_{11}-r_{22}-r_{33}+1}\ ;$$

where $\eta = \pm 1$

Lemma: For all rotations, at least one of the Euler Parameters has a magnitude larger than or equal to 1/2.

Algorithm

	$\lvert\lambda_0((t_{(i-1)})\rvert$	$\lvert\lambda_1(t_{(i-1)})\rvert$	$\lvert\lambda_2(t_{(i-1)})\rvert$	$\lvert\lambda_3(t_{(i-1)})\rvert$
$\lambda_0(t_i)$	$\Delta_0/4$	$(s_{32}-s_{23})/\Delta_1$	$(s_{13}-s_{31})/\Delta_2$	$(s_{21}-s_{12})/\Delta_3$
$\lambda_1(t_i)$	$(s_{32}-s_{23})/\Delta_0$	$\Delta_1/4$	$(s_{21}+s_{12})/\Delta_2$	$(s_{13}+s_{31})/\Delta_3$
$\lambda_2(t_i)$	$(s_{13}-s_{31})/\Delta_0$	$(s_{21}+s_{12})/\Delta_1$	$\Delta_2/4$	$(s_{32}+s_{23})/\Delta_3$
$\lambda_3(t_i)$	$(s_{21}-s_{12})/\Delta_0$	$(s_{13}+s_{31})/\Delta_1$	$(s_{32}+s_{23})/\Delta_2$	$\Delta_3/4$

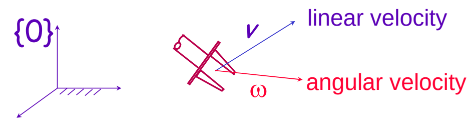
with

$$\begin{aligned} \Delta_0 &= 2\operatorname{sgn}\left(\lambda_0(t_{(i-1)})\right)\sqrt{s_{11}+s_{22}+s_{33}+1}; \\ \Delta_1 &= 2\operatorname{sgn}\left(\lambda_1(t_{(i-1)})\right)\sqrt{s_{11}-s_{22}-s_{33}+1}; \\ \Delta_2 &= 2\operatorname{sgn}\left(\lambda_2(t_{(i-1)})\right)\sqrt{-s_{11}+s_{22}-s_{33}+1}; \\ \Delta_3 &= 2\operatorname{sgn}\left(\lambda_3(t_{(i-1)})\right)\sqrt{-s_{11}-s_{22}+s_{33}+1}. \end{aligned}$$

Euler Angles & Parameters

$$\lambda_0 = \cos\theta/2 \cdot \cos(\psi+\phi)/2\ ;$$
$$\lambda_1 = \sin\theta/2 \cdot \cos(\psi-\phi)/2\ ;$$
$$\lambda_2 = \sin\theta/2 \cdot \sin(\psi-\phi)/2\ ;$$
$$\lambda_3 = \cos\theta/2 \cdot \sin(\psi+\phi)/2\ .$$

Basic Jacobian



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J_0(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

Jacobian for X

Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q)\ \dot{q}$$

$$J_x(q) = E(x)\ J_0(q)$$

Basic Jacobian $\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\ \dot{q}$

Jacobian and Basic Jacobian

$$J = \begin{pmatrix} J_{XP} \\ J_{XR} \end{pmatrix} = \left(\begin{array}{c|c} E_P & 0 \\ \hline 0 & E_R \end{array}\right) \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

$$J(q) = E(X)\ J_0(q)$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\ \dot{q}$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_P(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x\ \ y\ \ z)^T = (\rho\cos\theta\ \ \ \rho\sin\theta\ \ \ z)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x\ \ y\ \ z)^T = (\rho\cos\theta\sin\phi\ \ \ \rho\sin\theta\sin\phi\ \ \ \rho\cos\theta)^T$$

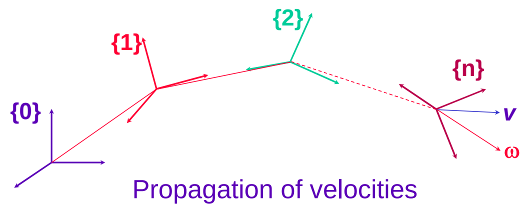
$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

Euler Angles

$$x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha . c\beta}{s\beta} & \frac{c\alpha . c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

Singularity of the representation for $\beta = k\pi$

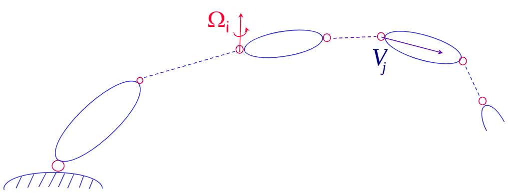
Spatial Mechanisms



Propagation of velocities

$\dot{x}$ — v : linear velocity; ω : angular velocity

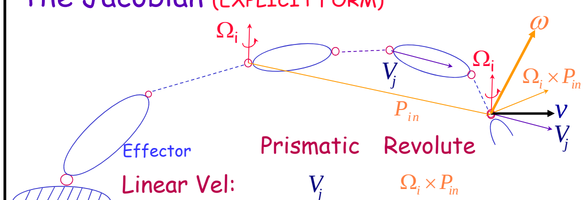
The Jacobian (EXPLICIT FORM)



Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

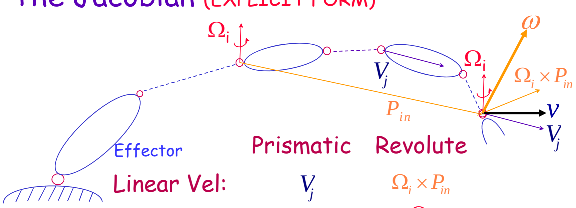
Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i Z_i + \bar{\epsilon}_i (Z_i \times P_{in})] \dot{q}_i \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} (\bar{\epsilon}_i Z_i) \dot{q}_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots$$
$$+ [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1} (Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) \quad \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) \quad \cdots] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = [\bar{\epsilon}_1 Z_1 \quad \bar{\epsilon}_2 Z_2 \quad \cdots \quad \bar{\epsilon}_n Z_n] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

The Jacobian

$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_P = \frac{\partial x_P}{\partial q_1}.\dot{q}_1 + \frac{\partial x_P}{\partial q_2}.\dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n}.\dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \end{pmatrix}$$

Jacobian in a Frame

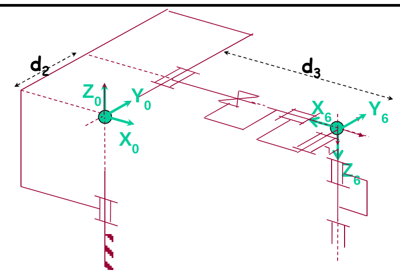
Vector Representation

$$J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \\ \bar{\epsilon}_1 . Z_1 & \bar{\epsilon}_2 . Z_2 & \cdots & \bar{\epsilon}_n . Z_n \end{pmatrix}$$

In {0}

$${}^0J = \begin{pmatrix} \frac{\partial\, {}^0x_P}{\partial q_1} & \frac{\partial\, {}^0x_P}{\partial q_2} & \cdots & \frac{\partial\, {}^0x_P}{\partial q_n} \\ \bar{\epsilon}_1 . {}^0Z_1 & \bar{\epsilon}_2 . {}^0Z_2 & \cdots & \bar{\epsilon}_n . {}^0Z_n \end{pmatrix}$$

Stanford Scheinman Arm



$$J = \begin{pmatrix} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & 0 & 0 \\ Z_1 & Z_2 & 0 & Z_4 & Z_5 & Z_6 \end{pmatrix}$$

Stanford Scheinman Arm Jacobian

$${}^0J = \begin{pmatrix} \frac{\partial\, {}^0x_P}{\partial q_1} & \frac{\partial\, {}^0x_P}{\partial q_2} & \frac{\partial\, {}^0x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix}$$

$$\begin{bmatrix} -c_1d_2 - s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ -s_1d_2 + c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1s_2 & -c_1c_2s_4 - s_1c_4 & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ 0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4 + c_1c_4 & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & -s_2c_4s_5 + c_5c_2 \end{bmatrix}$$

Velocity/Force Duality

$$\dot{x} = J\,\dot{\theta}$$

$$\tau = J^T F$$

Instantaneous Inverse Kinematics

Linearized Kinematic Model

$$\delta x = J(q)\,\delta q$$

Resolved Motion-Rate

(Whitney 1972)

$$\delta q = J^{-1}(q)\,\delta x$$

Jacobian

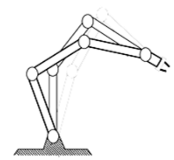
$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = J_{2\times 2}\begin{pmatrix}\Delta q_1\\ \Delta q_2\end{pmatrix}$$

Inverse Jacobian

$$\begin{pmatrix}\Delta q_1\\ \Delta q_2\end{pmatrix} = J^{-1}_{2\times 2}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}$$

Redundancy

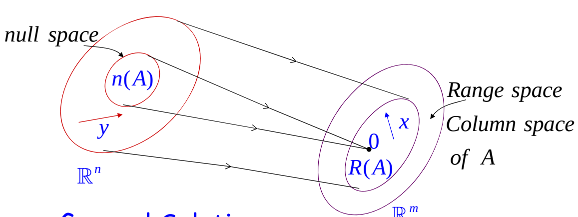
$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = J_{2\times 3}\begin{pmatrix}\Delta q_1\\ \Delta q_2\\ \Delta q_3\end{pmatrix}$$



Generalized Inverse

$$\begin{pmatrix}\Delta q_1\\ \Delta q_2\\ \Delta q_3\end{pmatrix} = J^{\#}_{3\times 2}\begin{pmatrix}\Delta x_1\\ \Delta x_2\end{pmatrix} + \left[I - J^{\#}J\right]_{3\times 3}\begin{bmatrix}\Delta q_1\\ \Delta q_2\\ \Delta q_3\end{bmatrix}$$

System $A_{(m\times n)}\,y_{(n\times 1)} = x_{(m\times 1)}$



General Solution

$$y = A^{\#}x + \left[I_n - A^{\#}A\right]y_0$$

Generalized Inverse

$A_{(m\times n)}$; $rank(A) = r$

$A^{\#}_{(n\times m)}$: $AA^{\#}A = A$

Example

$$A = (2 \quad -1)$$

$$A^{\#} = \begin{pmatrix}\frac{1}{2}+\frac{a}{2}\\ a\end{pmatrix}$$

Example

$$Ay = (2 \quad -1)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x$$

$$A^{\#}x = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix} x = \begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$Ay = (2 \quad -1)\begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = x$$

$$A_{(m\times n)} y_{(n\times 1)} = x_{(m\times 1)}$$

$\dfrac{n > m}{(r = m)}$ → Less equations than unknowns

→ Free variables

→ ∞ solutions

Example
$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

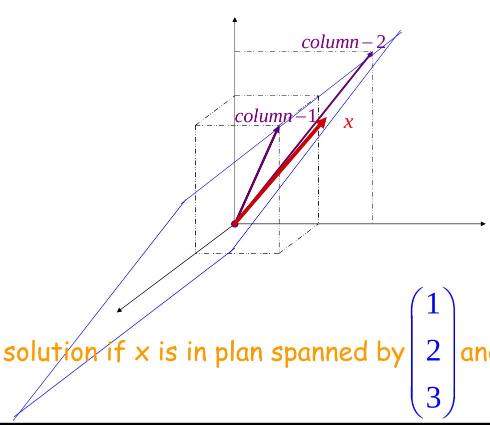
$\dfrac{n < m}{(r = n)}$ → More equations than unknowns

→ At most one solution

Example
$$\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$y_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y_2\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

solution if x is in plan spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

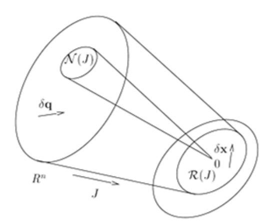


solution if x is in plan spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

Jacobian Generalized Inverse

Generalized Inverse

$$J_0^{\#}: \quad J_0 J_0^{\#} J_0 = J_0$$

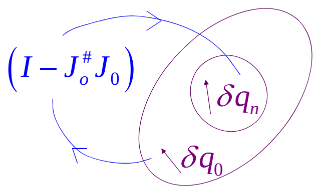


General Solution

$$\delta q = J_0^{\#}\,\delta x_0 + \left[I_n - J_0^{\#} J_0\right]\delta q_0$$

General Solution

$$\delta q = J_0^{\#}\,\delta x_0 + \underbrace{\left[I_n - J_0^{\#} J_0\right]\delta q_0}_{\delta q_n}$$



$$\delta q_n = \left(I - J_0^{\#} J_0\right)\delta q_0$$

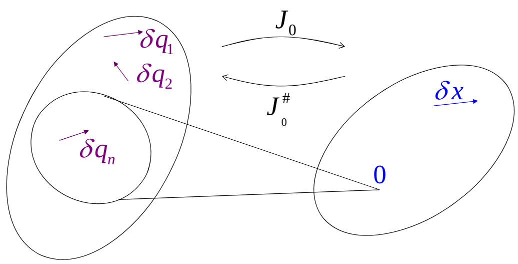
$$0 = J_0\,\delta q_n$$

$$\Rightarrow \quad 0 = J_0\left(I - J_0^{\#} J_0\right)\delta q_0$$

$$0 \equiv J_0 - J_0 J_0^{\#} J_0$$

$$\Rightarrow$$

$$J_0^{\#}: \quad J_0 \triangleq J_0 J_0^{\#} J_0$$



$$J_0^{\#}: J_0 = J_0 J_o^{\#} J_0$$

Pseudo Inverse

$$AA^+A = A$$

$$A^+AA^+ = A^+$$

$$\left(A^+A\right)^T = A^+A$$

$$\left(AA^+\right)^T = AA^+$$

A^+ : *unique*

Pseudo-Inverse

Left Inverse

$m > n$ $\quad A^+ = \left(A^T A\right)^{-1} A^T$

$(r = n)$ $\quad A^+A = I$

$m = n = r$ $\quad A^+ = A^{-1}$

$A^+A = AA^+ = I$

Right Inverse

$m < n$ $\quad A^+ = A^T (AA^T)^{-1}$

$(r = m)$ $\quad AA^+ = I$

Generalized Inverse

Left Inverse

$m > n$ $\quad A^{\#} = \left(A^T W^{-1} A\right)^{-1} A^T W^{-1}$

$(r = n)$ $\quad A^{\#}A = I$

$m = n = r$ $\quad A^{\#} = A^{-1}$

$A^{\#}A = AA^{\#} = I$

Right Inverse

$m < n$ $\quad A^{\#} = W^{-1}A^T (AW^{-1}A^T)^{-1}$

$(r = m)$ $\quad AA^{\#} = I$

Reduction to the Basic Kinematic Model

Initial Problem (m equations)

$$J\,\delta q = \delta x$$

Reduced Problem (m_0 equations) $J = E\,J_0$

$$\delta x = E(X)\,\delta x_0$$

$$J_0(q)\,\delta q = \delta x_0$$

Solving $\delta x = E(X)\,\delta x_0$

$E(X): m \times m_0$ matrix $(m \geq m_0)$

$-rank\left(E(X)\right) \leq m_0$

$-rank\left(E(X)\right) < m_0$ at configuration x where the representation is singular

Left Inverse

If $rank\left(E(X)\right) = m_0$ the system has a unique solution:

$$\delta x_0 = E^{+}_{(m_0 \times m)}(X)\;\delta x$$

E^{+}: is such that $E^{+}E = \mathrm{I}_{m_0}$

$$E^{+} = \left(E^{T}E\right)^{-1}E^{T}$$

and

$$E^{+}(X) = \begin{pmatrix} E_p^{+}(Xp) & 0 \\ 0 & E_r^{+}(Xr) \end{pmatrix}$$

System

$$\delta x_{m\times 1} = E_{m\times m_0}\,\delta x_{0\,m_0\times 1}$$

$$E^{T}_{m_0\times m}\,\delta x_{m\times 1} = \left(E^{T}E\right)_{m_0\times m_0}\,\delta x_{0\,m_0\times 1}$$

$$\left(E^{T}E\right)^{-1}E^{T}\,\delta x = \delta x_0$$

$$\delta x_0 = E^{+}\,\delta x$$

$$E^{+} = \left(E^{T}E\right)^{-1}E^{T}$$

$$E^{+}E = \underbrace{\left(E^{T}E\right)^{-1}E^{T}}_{\text{Left Inverse}}\,E = I$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_P(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \;\; y \;\; z)^T = (\rho\cos\theta \;\; \rho\sin\theta \;\; z)^T$

$$E_p(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_P^{+}(X) = E_P^{-1}(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ -\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ,θ,ϕ)

Using

$$(x\ \ y\ \ z)^T = (\rho\cos\theta\sin\phi\ \ \rho\sin\theta\sin\phi\ \ \rho\cos\theta)^T$$

$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

Spherical Coordinates (ρ,θ,ϕ)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ -\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix};\ E_r(x_r) = \begin{pmatrix} -\hat{S}_1 \\ -\hat{S}_2 \\ -\hat{S}_3 \end{pmatrix}$$

$$E_r^+ = \left(E_r^T E_r\right)^{-1} E_r^T$$

$$\left(E_r^T E_r\right)^{-1} = \left(\hat{S}_1^T\hat{S}_1 + \hat{S}_2^T\hat{S}_2 + \hat{S}_3^T\hat{S}_3\right)^{-1}$$

Example

$$S = \left(S_1 S_2 S_3\right) = \begin{pmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{z} = \widehat{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}} = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

$$\hat{S}_1 = \begin{bmatrix} 0 & 0 & S1 \\ 0 & 0 & -C1 \\ -S1 & C1 & 0 \end{bmatrix}$$

$$\hat{S}_1^T\hat{S}_1 = \begin{bmatrix} S_1^2 & -SC1 & 0 \\ -SC1 & C_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$SC1 = S1\ C1$

$$\hat{S}_2 = \begin{bmatrix} 0 & 0 & C1 \\ 0 & 0 & S1 \\ -C1 & -S1 & 0 \end{bmatrix}$$

$$\hat{S}_2^T\hat{S}_2 = \begin{bmatrix} C_1^2 & SC1 & 0 \\ SC1 & S_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{S}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{S}_3^T \hat{S}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E^T E = \sum \hat{S}_i^T \hat{S}_i = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I_3$$

$$\forall X_r = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \Rightarrow E^T E = \sum \hat{S}_i^T \hat{S}_i = 2I_3$$

$$\left(E_r^T E_r\right)^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \tfrac{1}{2} I_3$$

$$E_r^+ = \left(E_r^T E_r\right)^{-1} E_r^T = \frac{1}{2} E_r^T$$

$$E_r^+ = \tfrac{1}{2}\left(-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T\right)$$

$$E_r^+ = \tfrac{1}{2}\left(\hat{S}_1 \hat{S}_2 \hat{S}_3\right)$$

Angular Velocity

$$x_r = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}; \; E_r(x_r) = \begin{pmatrix} -\hat{s}_1 \\ -\hat{s}_2 \\ -\hat{s}_3 \end{pmatrix}$$

$$\dot{X}_r = E_r \omega$$

Solution

$$\omega = \frac{1}{2} E^T \dot{X}_r$$

Direction Cosines – Rotation Error

Instantaneous Angular Error

$$x_r = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}; \; x_{rd} = \begin{bmatrix} S_{1d} \\ S_{2d} \\ S_{3d} \end{bmatrix}$$

$$\delta x_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} - \begin{pmatrix} S1d \\ S2d \\ S3d \end{pmatrix}$$

$$\omega = \frac{1}{2} E^T \dot{X}_r$$

$$\delta\phi = \frac{1}{2} E^T \delta X_r$$

$$\delta X_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} - \begin{pmatrix} S1d \\ S2d \\ S3d \end{pmatrix}$$

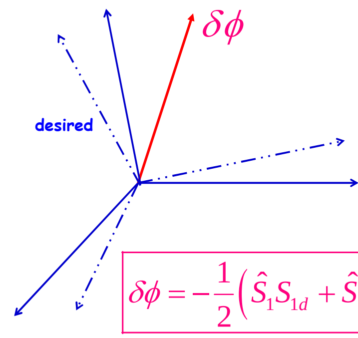
$$E_r^+ = \frac{1}{2}E^T$$

$$E^T = \left(-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T\right)$$

$$E_r^+ = \frac{1}{2}\left(\hat{S}_1\ \hat{S}_2\ \hat{S}_3\right)$$

$$E_r^+ \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} = \frac{1}{2}(\hat{S}_1 S1 + \hat{S}_2 S2 + \hat{S}_3 S3) \equiv 0$$

Instantaneous Angular Error



$$\delta\phi = -\frac{1}{2}\left(\hat{S}_1 S_{1d} + \hat{S}_2 S_{2d} + \hat{S}_3 S_{3d}\right)$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix};\ E_r(x_r) = \begin{pmatrix} -\hat{S}_1 \\ -\hat{S}_2 \\ -\hat{S}_3 \end{pmatrix}$$

Observing $E_r^T(x_r)E_r(x_r) = 2I_3$

$$E_r^+(x_r) = \frac{1}{2}\left(-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T\right)$$

Euler Angles

$$E_r(X) = \begin{pmatrix} -{}^{S\varphi C\theta}/_{S\theta} & {}^{C\varphi C\theta}/_{S\theta} & 1 \\ C\varphi & S\varphi & 0 \\ {}^{S\varphi}/_{S\theta} & -{}^{C\varphi}/_{S\theta} & 0 \end{pmatrix}$$

$$E_r^{-1}(x_r) = \begin{pmatrix} 0 & \cos\psi & \sin\psi\sin\theta \\ 0 & \sin\psi & -\cos\psi\sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}$$

Euler Parameters

$$x_r = \lambda = \left(\lambda_0\ \lambda_1\ \lambda_2\ \lambda_3\right)^T$$

$$\dot{\lambda} = \frac{1}{2}\check{\lambda}\,\omega$$

$$\check{\lambda} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

$$E_r(\lambda) = \frac{1}{2}\check{\lambda}$$

Euler Parameters

Observing

$$\check{\lambda}^T \check{\lambda} = I_3$$

$$E_r^+(x_r) = 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

Inverse of the Basic Kinematic Model

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{(m_0\times n)}\delta q_{(n\times 1)} \; ; \; m_o \le n$$

Right Inverse

A solution iff $rank \ J_0 = m_0$

$\exists$ an $n \times m_0$ right inverse $J_0^{\#} / J_0 J_0^{\#} = I_{m_0}$

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{(m_0\times n)}\delta q_{(n\times 1)} \; ; \; m_o \le n$$

Solution $\delta q = J_0^{\#}\, \delta X_0$

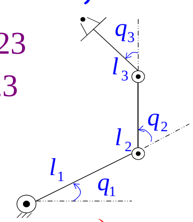
$J_0^{\#}$: Generalized Inverse

General Solution

$$\delta q = J_0^{\#}\, \delta X_0 + \underbrace{\left[I_n - J_0^{\#} J_0 \right] \delta q_0}_{\delta q_n}$$

Redundancy (w.r.t a Task)

$$x = l_1 C1 + l_2 C12 + l_3 C123$$
$$y = l_1 S1 + l_2 S12 + l_3 S123$$



$$q_1 = q_2 = 0$$
$$l_1 = l_2 = l_3 = 1$$

$$J_{2\times 3}(q) = \begin{pmatrix} -S3 & -S3 & -S3 \\ 2+C3 & 1+C3 & C3 \end{pmatrix}$$

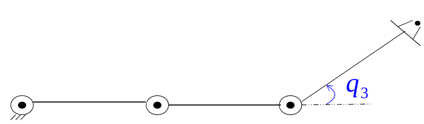
$$J^+_{3\times 2} = J^T \left(J\, J^T \right)^{-1}$$

$$J\, J^T = \begin{pmatrix} 3S_3^2 & -3(1+C3)S3 \\ -3(1+C_3)S3 & 3C_3^2 + 6C3 + 5 \end{pmatrix}$$

$$\det\left(J\, J^T \right) = 6S_3^2$$

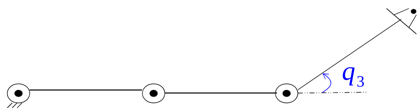
$$\left(J\, J^T \right)^{-1} = \frac{1}{6S_3^2}\begin{pmatrix} 3C_3^2 + 6C3 + 5 & 3(1+C3)S3 \\ 3(1+C_3)S3 & 3S_3^2 \end{pmatrix}$$

$$J^+_{(3\times 2)} = \frac{1}{6S3}\begin{pmatrix} 1+3C3 & 3S3 \\ -2 & 0 \\ -(5+3C3) & -3S3 \end{pmatrix}$$



$$\delta q = J^+\, \delta x + \left(I_n - J_0^+ J_0 \right) \delta q_0$$

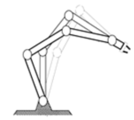
$$J^+ J = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$



$$\delta q = J^{+}\,\delta x + \left(I_n - J_0^{+} J_0\right)\,\delta q_0$$

$$\left(I - J^{+}\,J\right) = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\delta q_n = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}\delta q_0$$



$$\delta q_{n1} = \frac{1}{6}\left(\delta q_1 - 2\delta q_2 + \delta q_3\right)$$

$$\delta q_{n2} = \frac{1}{6}\left(-2\delta q_1 + 4\delta q_2 - 2\delta q_3\right)$$

$$\delta q_{n3} = \frac{1}{6}\left(\delta q_1 - 2\delta q_2 + \delta q_3\right)$$

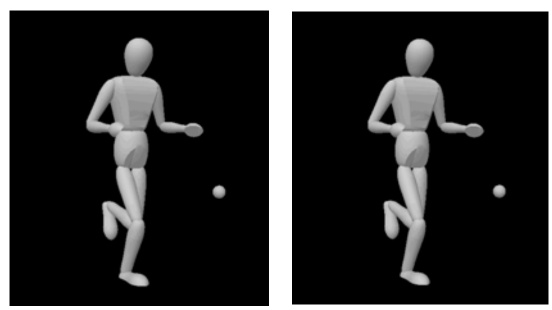
Redundancy

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{\,(m_0\times n)}\,\delta q_{(n\times 1)}\;;\; m_o \le n$$

General Solution

$$\delta q = J_0^{\#}\,\delta X_0 + \underbrace{\left[I_n - J_0^{\#} J_0\right]\delta q_0}_{\delta q_n}$$



Kinematic Singularity

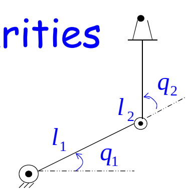
The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \left(J_1\ J_2\ \cdots\ J_n\right)$$

$$\det\left(J\right) = 0$$

$$\det\left({}^{i}J\right) = \det\left({}^{j}J\right)$$

Kinematic Singularities



$$x = l_1 C1 + l_2 C12$$

$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -\left(l_1 S1 + l_2 S12\right) & -l_2 S12 \\ l_1 C1 + l_2 S12 & l_2 C12 \end{pmatrix}$$

$$\det\left(J\right) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

$$J = S_{01} J_{(1)}$$

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}$$

At Singularity

$$J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

The rank of $(J\,J^T) < 2$

$(J^T J) < 2$

Singular Value Decomposition

Theorem - Definition

Any m x n matrix A of rank r can be factored into:

$A = U \sum V^T$; where

$-U$ is an m x m orthogonal matrix;

$-V$ is an n x n orthogonal matrix;

$-\sum$ is an m x n matrix of the form

$$\Sigma = \left(\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array} \right)$$; with

$\Sigma_r = diag[\sigma_i]$ with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

$\sigma_i (i = 1, \cdots, r)$ are uniquely determined for A and called "Singular values of A"

Decomposition of A

$$A_{(m \times n)} = U_{(m \times m)} \Sigma_{(m \times n)} V^T{}_{(n \times n)}$$

$m \geq n$

$$A^T A = V \left(\Sigma^T \Sigma \right) V^T$$

$$\Sigma^T \Sigma = \left(\begin{array}{c|c} \Sigma_r^2 & 0 \\ \hline 0 & 0 \end{array} \right)$$

1^o) $\det\left(A^T A - \sigma^2 I\right) = 0 \rightarrow \sigma_1^2, \cdots, \sigma_r^2 \rightarrow \Sigma$

2^o) $\left(A^T A - \sigma_i^2 I\right) V_i = 0 \rightarrow V$

3^o) $AV = U \Sigma$

$$\left(Av_1 \vdots Av_2 \vdots \cdots\right) = \left(\sigma_1 u_1 \vdots \sigma_2 u_2 \vdots \cdots\right)$$

$$u_i = \frac{Av_i}{\sigma_i} \rightarrow U$$

$m < n$

$$A A^T = U \left(\Sigma^T \Sigma \right) U^T$$

$$\left(\Sigma \Sigma^T \right)_{(m \times m)} = \left(\begin{array}{c|c} \Sigma_r^2 & 0 \\ \hline 0 & 0 \end{array} \right)$$

$1^o)\ \det\left(AA^T - \sigma^2 I\right) = 0 \rightarrow \Sigma$

$2^o)\left(AA^T - \sigma_i^2 I\right)U_i = 0 \rightarrow U$

$3^o)\ A^T U = V\Sigma$

$$\left(A^T u_1 \vdots A^T u_1 \vdots \cdots\right) = \left(\sigma_1 v_1 \vdots \sigma_2 v_2 \vdots \cdots\right)$$

$$v_i = \frac{A^T u_i}{\sigma_i} \rightarrow V$$

Pseudo Inverse of $A = U\,\Sigma\,V^T$ is

$$\underline{\underline{A^+ = V\,\Sigma^+ U^T}}$$

$$AA^+A = A$$

$$\left(A^+A\right)^T = A^+A$$

$$\left(AA^+\right)^T = AA^+$$

Pseudo Inverse of $\Sigma = \begin{pmatrix} \sigma_i & & 0 \\ 0 & \sigma_r & \\ & & 0 \end{pmatrix}$ is

$$\Sigma^+ = \begin{pmatrix} \frac{1}{\sigma_i} & & 0 \\ 0 & \frac{1}{\sigma_i} & \\ & & 0 \end{pmatrix}$$

Example

$$J_{(l)} = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

$1^o)\ \Sigma?$

$$J^T J = \begin{pmatrix} \left(l_1 + l_2\right)^2 & \left(l_1 + l_2\right)l_2 \\ \left(l_1 + l_2\right)l_2 & l_2^2 \end{pmatrix}$$

$$\det\left(J^T J - \sigma^2 I\right) = 0$$

$$\sigma_1^2 = l_2^2 + \left(l_1 + l_1\right)^2$$

$$\sigma_2^2 = 0$$

$$\Sigma = \begin{pmatrix} \sqrt{l_2^2 + \left(l_1 + l_2\right)^2} & 0 \\ 0 & 0 \end{pmatrix}$$

$2^o)\ V?$

$$\left(J^T J - \sigma_1^2 I\right)v_1 = 0$$

$$\begin{pmatrix} -l_2^2 & \left(l_1 + l_2\right)l_2 \\ \left(l_1 + l_2\right)l_2 & -\left(l_1 + l_2\right)^2 \end{pmatrix} v_1 = 0$$

$$V = \frac{1}{\sqrt{l_2^2 + \left(l_1 + l_2\right)}} \begin{pmatrix} l_1 + l_2 & -l_2 \\ l_2 & l_1 + l_2 \end{pmatrix}$$

$3^o)\ U\,?$

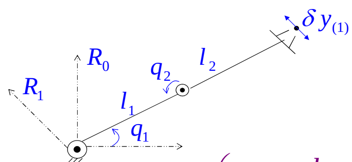
$$u_i = \frac{J v_i}{\sigma_i}$$

$$u_1 = \begin{pmatrix} 0 & 0 \\ \frac{l_1+l_2}{\sqrt{(.)}} & \frac{l_2}{\sqrt{(.)}} \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2}{\sqrt{(.)}} \\ \frac{l_2}{\sqrt{(.)}} \end{pmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

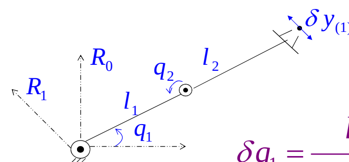
$$J = S_{01}\ U\ \Sigma\ V^T$$



$$\delta x_{(1)} = J_{(1)} \delta q$$

$$\delta q = J^+_{(1)}\ \delta x_{(1)}$$

$$\delta q = \begin{pmatrix} 0 & \frac{l_1+l_2}{l_2^2 (l_1+l_2)^2} \\ 0 & \frac{l_2}{l_2^2 (l_1+l_2)^2} \end{pmatrix} \delta x$$



$$\delta q_1 = \frac{l_1+l_2}{l_2^2 (l_1+l_2)^2}\ \delta y_{(1)}$$

$$\delta q_2 = \frac{l_2}{l_2^2 (l_1+l_2)^2}\ \delta y_{(1)}$$

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{l_2^2 + (l_1+l_2)^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2}{\sqrt{(.)}} & \frac{l_2}{\sqrt{(.)}} \\ \frac{-l_2}{\sqrt{(.)}} & \frac{l_1+l_2}{\sqrt{(.)}} \end{pmatrix}$$

$$J^+ = V\ \Sigma^+\ U^T\ S_{01}^T$$

$$J^+ = \begin{pmatrix} 0 & \frac{l_1+l_2}{l_2^2 + (l_1+l_2)^2} \\ 0 & \frac{l_2}{l_2^2 + (l_1+l_2)^2} \end{pmatrix} \begin{pmatrix} C1 & S1 \\ -S1 & C1 \end{pmatrix}$$

General Solution

$$\delta q = J^+\ \delta x + \underbrace{\left(I - J^+ J\right)\ \delta q_0}_{\delta q_n}$$

$$J^+ J = \frac{1}{\sqrt{l_2^2 + (l_1+l_2)^2}} \begin{pmatrix} l_1+l_2 & -l_2 \\ l_2 & l_1+l_2 \end{pmatrix}$$

$$\delta q_n = \frac{1}{l_2^2 + (l_1+l_2)^2} \begin{pmatrix} l_2^2 & -(l_1+l_2)l_2 \\ -(l_1+l_2)l_2 & (l_1+l_2)^2 \end{pmatrix} \delta q_0$$

$$\delta x_n = J\,\delta q_n = 0$$

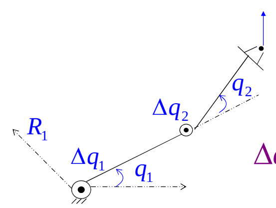
Problem with the Pseudo Inverse Solution

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \underbrace{\begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}}_{J_{(1)}}$$

$$J^{-1} = \underbrace{\frac{1}{l_1 l_2 S2} \begin{pmatrix} l_2 C2 & l_2 S2 \\ -(l_1 + l_2 C2) & -l_2 S2 \end{pmatrix}}_{J_{(1)}^{-1}} \begin{pmatrix} C1 & S1 \\ -S1 & C1 \end{pmatrix}$$

small θ_2

$$J_{(1)}^{-1} \cong \begin{pmatrix} \frac{1}{l_1 \theta_2} & \frac{1}{l_1} \\ -\frac{l_1 + l_2}{l_1 l_2 \theta_2} & -\frac{1}{l_1} \end{pmatrix}$$

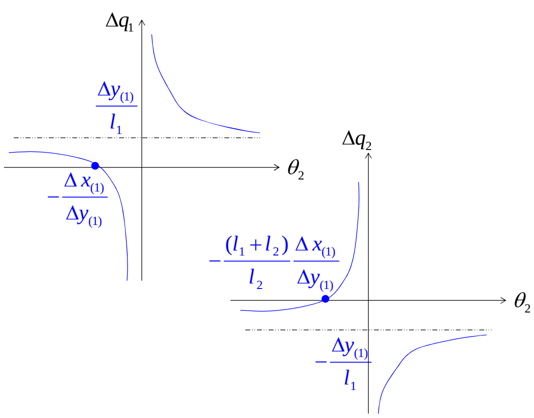


$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

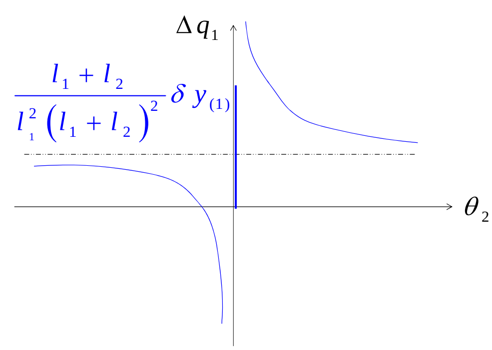
$$\Delta q = J^{-1} \Delta X$$

$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{(l_1 + l_2)\Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



Pseudo Inverse Solution



Singularity Robust Inverse

Pseudo-Inverse

$$J^{+} = J^{T}\left(J\,J^{T}\right)^{-1}$$

S-R Inverse

$$J^{*} = J^{T}\left(J\,J^{T} + k\,I\right)^{-1}$$

Singularity Robust Inverse