

### Efficient Dynamic Algorithms

*Simulation, Contact Resolution, and Control*

- Efficient algorithms for contact dynamics  $O(n)$
- Avoid constraints elimination
- Cost-free effective mass at arbitrary contacts

### Human-Friendly Robots

- Safety
- Performance

**Competing!**

### Effective Inertia

2 DOF Example

Effective Inertia at End-Effector

$$J_{\text{effective}} = (J_{\text{link}} + N^2 J_{\text{motor}})$$

### Effective Mass/Inertia

#### Task Redundancy

### Redundancy

- Redundancy of a Manipulator
 
$$n > m_0$$
- Redundancy with respect to a Task
 
$$n > m_{\text{Task}(0)}$$

### Task dynamics

$$\Lambda(q)\ddot{x} + \mu(q, \dot{q}) + p(q) = F$$

$$\Lambda = (JA^{-1}J^T)^{-1}$$

$$\mu(q, \dot{q}) = \bar{J}^T b(q, \dot{q}) - \Lambda(q)\dot{J}(q)\dot{q}$$

$$p(q) = \bar{J}^T g(q)$$

### Task Redundancy

$$\dot{y} = \begin{pmatrix} \frac{\partial y}{\partial q_1} & \dots & \frac{\partial y}{\partial q_n} \end{pmatrix} \dot{q}$$

$$\dot{y} = J_{v_y} \dot{q}$$

$$m_y = \frac{1}{J_{v_y} A^{-1} J_{v_y}^T}$$

$$I_z = \frac{1}{J_{\omega_z} A^{-1} J_{\omega_z}^T}$$

(a)

$$J_{v_y} = y_0^T J_v$$

(b)

$$J_{\omega_z} = z_0^T J_{\omega}$$

$$J_v = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \dots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \dots & \frac{\partial y}{\partial q_n} \\ \frac{\partial z}{\partial q_1} & \dots & \frac{\partial z}{\partial q_n} \end{pmatrix}$$

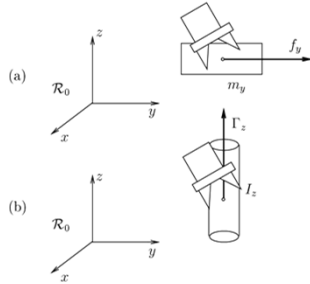
$J_{v_y} = y_0^T J_v$

$$J_{v_y} = (0 \ 1 \ 0) J_v = \begin{pmatrix} \frac{\partial y}{\partial q_1} & \dots & \frac{\partial y}{\partial q_n} \end{pmatrix}$$

## Effective Mass/Inertia

$$m_y = \frac{1}{y_0^T \Lambda_v^{-1} y_0}$$

$$I_z = \frac{1}{z_0^T \Lambda_\omega^{-1} z_0}$$



## Effective Mass/Inertia

$$J_{v_u} = u^T J_v$$

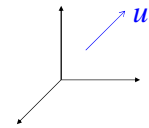
$$J_{\omega_u} = u^T J_\omega$$

Effective Mass

$$m_u(\Lambda_v) = \frac{1}{u^T \Lambda_v^{-1} u}$$

Effective Inertia

$$I_u(\Lambda_\omega) = \frac{1}{u^T \Lambda_\omega^{-1} u}$$



## Effective Mass/Inertia

$$\Lambda_0^{-1} = J_0 A^{-1} J_0^T$$

$$J_0 = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix}$$

$$\Lambda_0^{-1} = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix} A^{-1} \begin{pmatrix} J_v^T & J_\omega^T \end{pmatrix}$$

## Effective Mass/Inertia

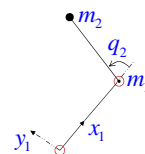
$$\Lambda_0^{-1} = \begin{bmatrix} \underbrace{J_v A^{-1} J_v^T}_{\Lambda_v^{-1}} & \underbrace{J_v A^{-1} J_\omega^T}_{\Lambda_\omega^{-1}} \\ \underbrace{J_\omega A^{-1} J_v^T}_{\Lambda_\omega^{-1}} & \underbrace{J_\omega A^{-1} J_\omega^T}_{\Lambda_\omega^{-1}} \end{bmatrix}$$

## Effective Inertial Property

$$\sigma_w(\Lambda) = \frac{1}{w^T \Lambda^{-1} w}$$

$w$ : A unit vector in the  $m_0$ -dimensional space

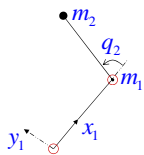
## Example



$$J_{(1)} = \begin{pmatrix} -S_2 & -S_2 \\ 1+C_2 & C_2 \end{pmatrix}$$

$$A = \begin{pmatrix} m_1 + 2m_2(1+C_2) & m_2(1+C_2) \\ m_2(1+C_2) & m_2 \end{pmatrix}$$

### Redundancy



with respect to motions  
in the  $y_1$  direction

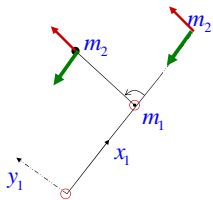
$$J_{y_1} = \begin{bmatrix} 1+C2 & C2 \end{bmatrix}$$

$$\Lambda_{y_1}^{-1} = J_{y_1}^{-1} A^{-1} J_{y_1}^T$$

$$\Lambda_{y_1}^{-1} = \frac{1}{m_2(m_1+m_2S^2)} \begin{bmatrix} 1+C2 & C2 \\ -m_2(1+C2) & m_1+2m_2(1+C2) \end{bmatrix} \begin{bmatrix} 1+C2 \\ C2 \end{bmatrix}$$

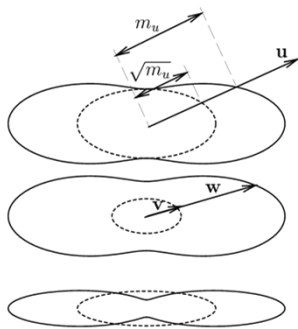
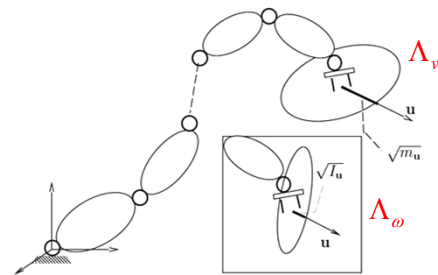
$$\frac{1}{m_{y_1}^*} = \frac{m_1C^2 + m_2S^2}{m_2(m_1+m_2S^2)}$$

$$m_{y_1}^* = \frac{m_2(m_1+m_2S^2)}{m_1C^2 + m_2S^2}$$



$$m_{x_1}^* = \frac{m_2(m_1+m_2S^2)}{(m_1+m_2)S^2}$$

### Effective Mass & Inertia

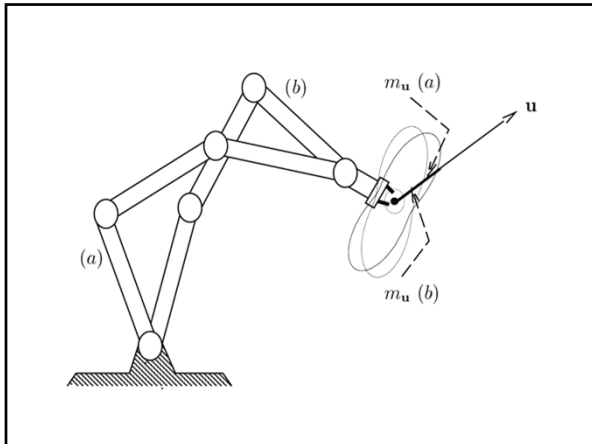


### Belted Ellipsoid

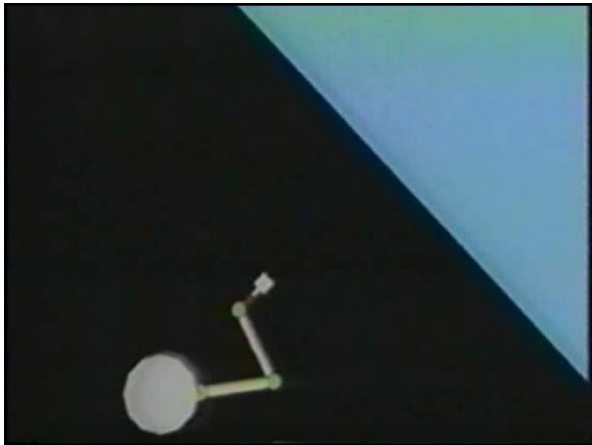


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

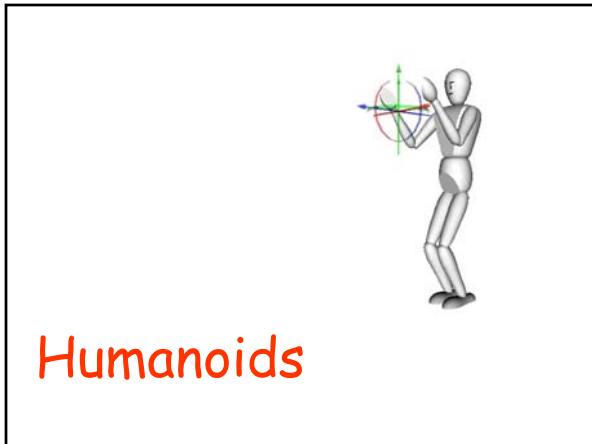
$$\Rightarrow \frac{x^2}{a^2\sqrt{x^2+y^2+z^2}} + \frac{y^2}{b^2\sqrt{x^2+y^2+z^2}} + \frac{z^2}{c^2\sqrt{x^2+y^2+z^2}} = 1$$



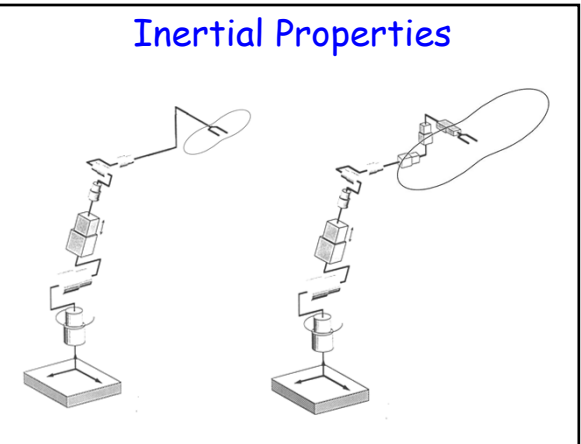
## Macro/Mini Structures

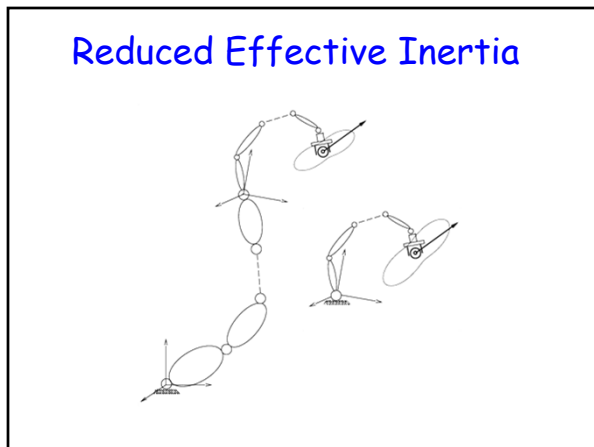
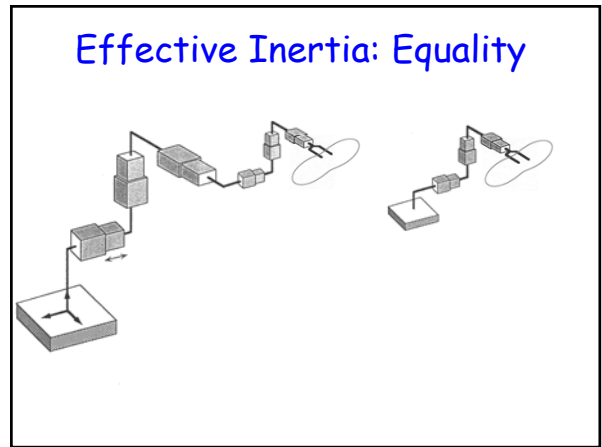
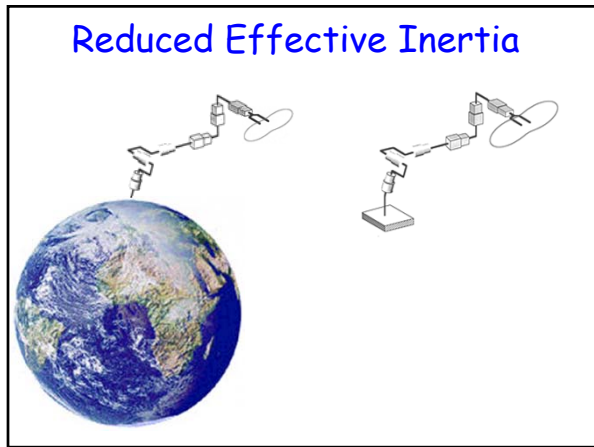
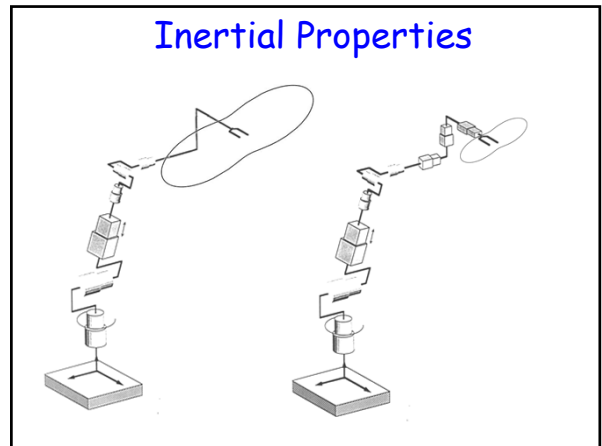
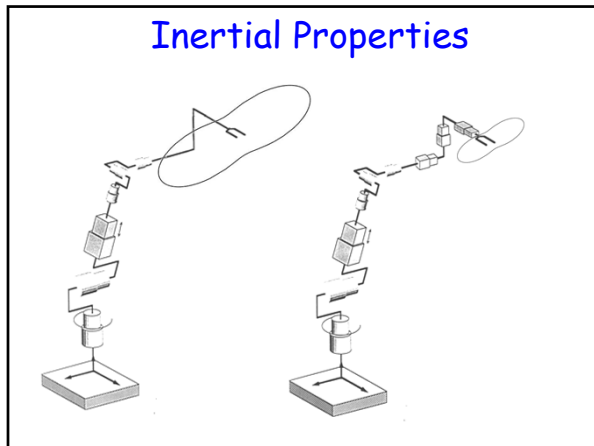


## Mobile Manipulators



## Humanoids





### Inertia Property

Effective Inertia/mass  
perceived in a direction  $\mathbf{w}$

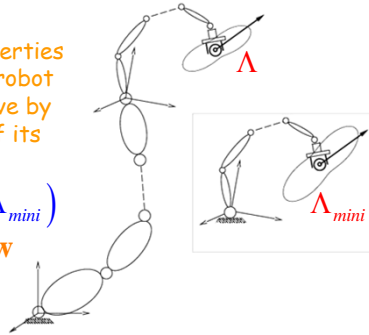
$$\sigma_{\mathbf{w}}(\Lambda) = \frac{1}{\mathbf{w}^T \Lambda^{-1} \mathbf{w}}$$

### Theorem: Reduced Effective Inertia

The inertial properties of a macro/mini robot are bounded above by the properties of its mini structure

$$\sigma_w(\Lambda) \leq \sigma_w(\Lambda_{mini})$$

in any direction  $w$



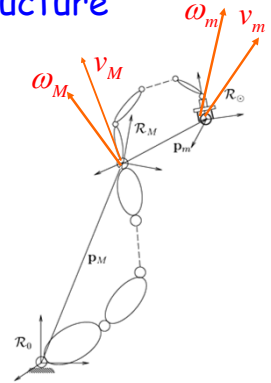
### Macro/Mini Structure

$$v = v_M + v_m + \omega_M \times p_m$$

$$\omega = \omega_M + \omega_m$$

$$J_0 = [V \ J_M \ | \ J_m]$$

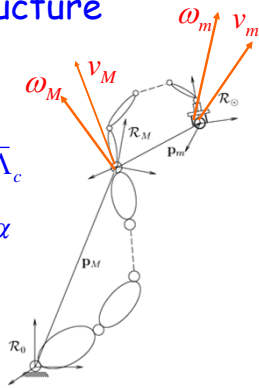
$$V = \begin{bmatrix} I_3 & -\hat{p}_m \\ 0 & I_3 \end{bmatrix}$$



### Macro/Mini Structure

- $\Lambda^{-1} = \Lambda_m^{-1} + \bar{\Lambda}_c$
- $\frac{1}{\sigma_w(\Lambda)} = \frac{1}{\sigma_w(\Lambda_m)} + \alpha$

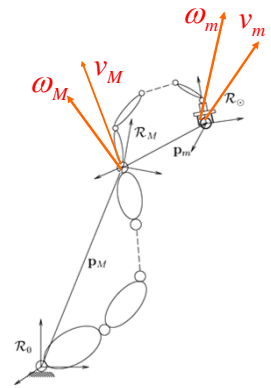
- $\alpha = w^T \bar{\Lambda}_c w$



A and A\_m

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

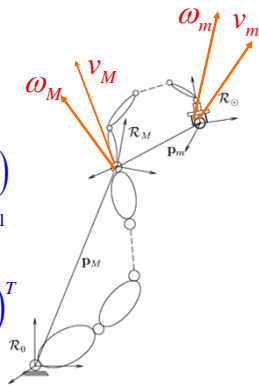
$$A_m = A_{22}$$



Lambda and Lambda\_m

$$\Lambda^{-1} = \Lambda_m^{-1} + \bar{\Lambda}_c$$

$$\bar{\Lambda}_c = \begin{pmatrix} VJ_M - J_m A_m^{-1} A_{21} \\ (A_{11} - A_{21}^T A_m^{-1} A_{21})^{-1} \\ (VJ_M - J_m A_m^{-1} A_{21})^T \end{pmatrix}$$



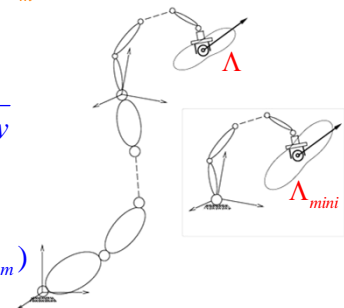
sigma\_w(Lambda) and sigma\_w(Lambda\_m)

$$\sigma_w(\Lambda) = \frac{1}{w^T \Lambda^{-1} w}$$

Theorem

$$\sigma_w(\Lambda) \leq \sigma_w(\Lambda_m)$$

for any  $w$



Corollary

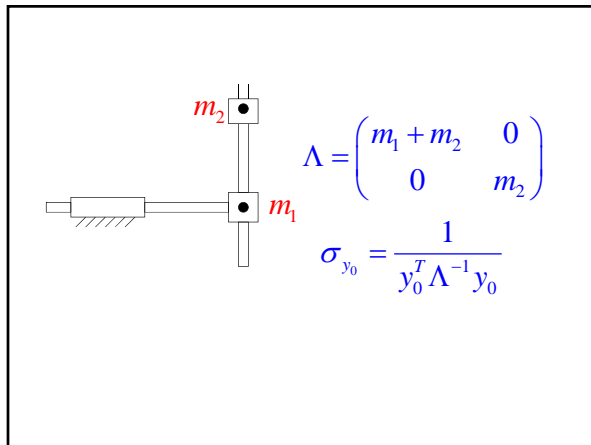
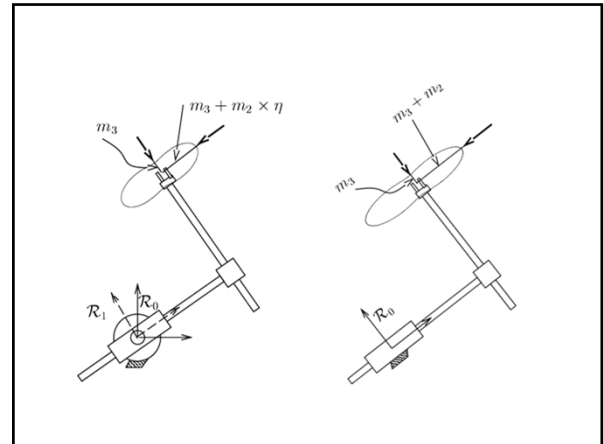
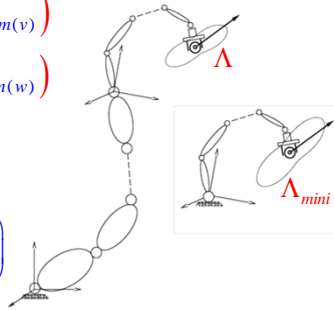
$$m_u(\Lambda_v) \leq m_u(\Lambda_{m(v)})$$

$$I_u(\Lambda_w) \leq I_u(\Lambda_{m(w)})$$

$$J_v = [I \quad 0] J$$

$$J_\omega = [0 \quad I] J$$

$$\bar{\Lambda}_{c(v)} = (I \quad 0) \bar{\Lambda}_c \begin{pmatrix} I \\ 0 \end{pmatrix}$$



$$\Lambda^{-1} = \begin{pmatrix} \frac{1}{m_1 + m_2} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}$$

$$\frac{1}{\sigma_{y_0}} = (0 \quad 1) \begin{pmatrix} \frac{1}{m_1 + m_2} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{m_2}$$

Kinetic energy matrix,  $\Lambda_m$  associated with the 2-degree-of-freedom mini-manipulator

$$\Lambda_m = \begin{bmatrix} m_2 + m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$J_{(1)} = \begin{bmatrix} -q_3 & 1 & 0 \\ q_2 & 0 & 1 \end{bmatrix}$$

Joint space kinetic energy matrix

$$A(q) = \begin{bmatrix} I + m_1 q_2^2 + m_2 (q_2^2 + q_3^2) & -q_3 m_2 & q_2 m_2 \\ -m_2 q_3 & m_1 + m_2 & 0 \\ m_2 q_2 & 0 & m_2 \end{bmatrix}$$

Kinetic energy matrix,  $\Lambda_{0(1)}$  associated with the 3-degree-of-freedom macro-/mini-manipulator

$$\Lambda_{(1)} = \begin{bmatrix} m_2 + m_1 \times \eta & 0 \\ 0 & m_2 \end{bmatrix}$$

where

$$\eta = \frac{I + m_1 q_2^2}{I + m_1 (q_2^2 + q_3^2)}$$

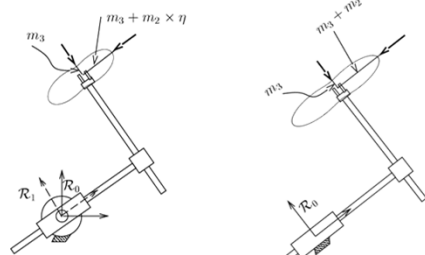
Kinetic energy matrix,  $\Lambda_0$

$$\Lambda = \Omega \Lambda_{(1)} \Omega^T$$

where

$$\Omega = \begin{bmatrix} C1 & -S1 \\ S1 & C1 \end{bmatrix}$$

with  $S1 \doteq \sin(q_1)$  and  $C1 \doteq \cos(q_1)$



$\Lambda_{(1)} = \begin{bmatrix} m_3 + m_2 \times \eta & 0 \\ 0 & m_3 \end{bmatrix}$ 
 $\Lambda_m = \begin{bmatrix} m_2 + m_3 & 0 \\ 0 & m_3 \end{bmatrix}$

$0 \leq \eta \leq 1; \quad \eta = \frac{I + m_2 q_2^2}{I + m_2 (q_2^2 + q_3^2)}$

