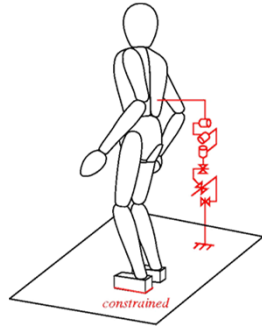
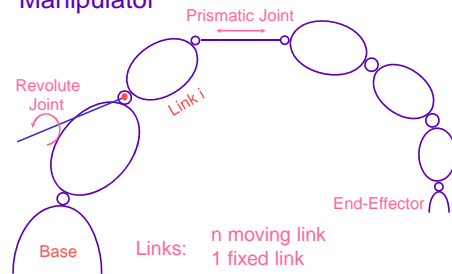


Kinematics



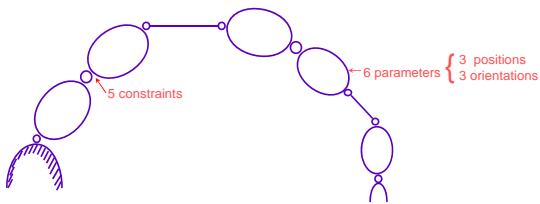
Manipulator



Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

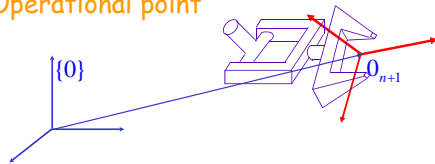
Generalized Coordinates



n moving links: 6n parameters
n 1 d.o.f. joints: 5n constraints
d.o.f. (system): $6n - 5n = n$

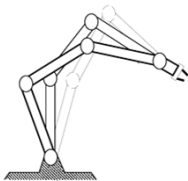
Operational Coordinates

O_{n+1} : Operational point



A set x_1, x_2, \dots, x_{m_0}
of m_0 independent configuration parameters
 m_0 : number of degrees of freedom
of the end-effector.

Redundancy



A robot is said to be redundant if

$$n > m_0$$

Degrees of redundancy: $n - m_0$

Task Redundancy



$$n > m_{task}$$

$n - m_{task}$: degrees of redundancy/task

Geometric Model

Homogeneous Transformation

→ Compact representation

$$\underline{x} = R\underline{x}' + \rho$$

$$\begin{pmatrix} \underline{x} \\ 1 \end{pmatrix} = \begin{pmatrix} R & \rho \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{x}' \\ 1 \end{pmatrix}$$

$$\underline{x} = T \underline{x}' \quad ; \quad T_{4 \times 4} = \begin{pmatrix} R & \rho \\ 0 & 1 \end{pmatrix}$$

→ T: not orthonormal

$$T^{-1} = \begin{pmatrix} R^T & -R^T \rho \\ 0 & 1 \end{pmatrix}$$

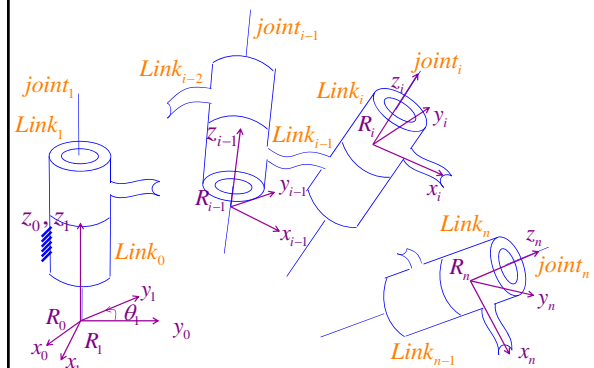
$$\underline{x} = T^{-1} \underline{x}'$$

Consecutive Transformations

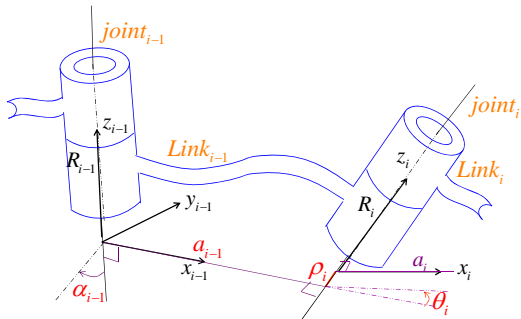
$$T_{1k} = T_1 T_2 \dots T_k$$

- Forward Kinematics
- Inverse Kinematics

Kinematic Chain



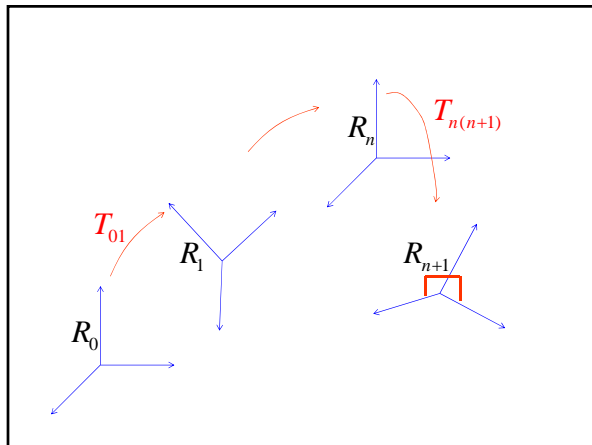
Denavit-Hartenberg (DH) Parameters



Homogeneous Transformation

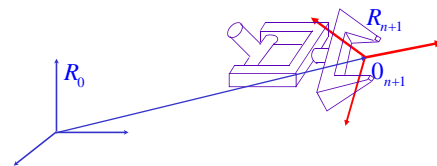
$$T_{(i-1)i} = T_{(i-1)i}(\alpha_{i-1}, a_{i-1}, \theta_i, \rho_i)$$

$$T_{(i-1)i} = \begin{pmatrix} \cos \theta_i & -\sin \theta_i & 0 & a_{(i-1)} \\ \sin \theta_i \cos \alpha_{(i-1)} & \cos \theta_i \cos \alpha_{(i-1)} & -\sin \alpha_{(i-1)} & -\rho_i \sin \alpha_{(i-1)} \\ \sin \theta_i \sin \alpha_{(i-1)} & \cos \theta_i \sin \alpha_{(i-1)} & \cos \alpha_{(i-1)} & \rho_i \cos \alpha_{(i-1)} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Geometric Model Forward Kinematics

$$T_{0(n+1)}(q) = T_{01}(q_1)T_{12}(q_2) \dots T_{(n-1)n}(q_n)T_{n(n+1)}$$



$$T_{0(n+1)}(q) = \begin{pmatrix} R_{o(n+1)}(q) & \rho_{o(n+1)}(q) \\ 0 & 1 \end{pmatrix}$$

m equations

$$x = G(q)$$

$$x = \begin{pmatrix} x_p(q) \\ x_r(q) \end{pmatrix}$$

Representations

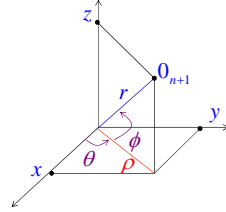
$$x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

- Cartesian
- Spherical
- Cylindrical
-
- Euler Angles
- Direction Cosines
- Euler Parameters

Position Representations

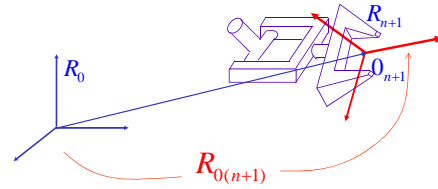
$x_p(q)$ is obtained from $\rho_{0(n+1)}(q)$

- Cartesian (x, y, z)
- Cylindrical (ρ, θ, z)
- Spherical (r, θ, ϕ)



Orientation Representation

$x_R(q)$ is obtained from $R_{0(n+1)}(q)$



$$R_{0(n+1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Direction Cosines

$$R_{0(n+1)} = (r_1(q) \ r_2(q) \ r_3(q));$$

$$x_r = \begin{pmatrix} r_1(q) \\ r_2(q) \\ r_3(q) \end{pmatrix}$$

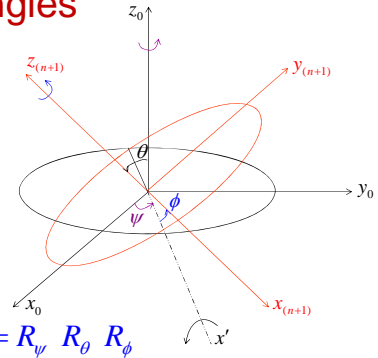
Euler Angles

$\psi/z_0 \rightarrow x'$

$\theta/x' \rightarrow z_{(n+1)}$

$\phi/z_{(n+1)}$

$$R_{0(n+1)} = R_\psi R_\theta R_\phi$$



$$R_{0(n+1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$R_{0(n+1)}(q) = \begin{pmatrix} c\psi c\phi - s\psi c\theta s\phi & -c\psi s\phi - s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi - c\psi c\theta s\phi & -s\psi s\phi + c\psi c\theta c\phi & -c\psi s\theta \\ s\theta s\phi & s\theta c\phi & c\theta \end{pmatrix}$$

$$x_r = \begin{pmatrix} \psi(q) \\ \theta(q) \\ \phi(q) \end{pmatrix}$$

$$\psi(q) = \text{sgn}(r_{13}) \arccos\left(-r_{23}/\sqrt{1-r_{33}^2}\right);$$

$$\theta(q) = \arccos(r_{33});$$

$$\phi(q) = \text{sgn}(r_{31}) \arccos\left(-r_{32}/\sqrt{1-r_{33}^2}\right);$$

$$r_{33} \neq \pm 1$$

Singularity of the representation

$$\theta = k\pi \quad k:(\text{integer}) \Rightarrow r_{33} \neq \pm 1$$

$(\psi + \phi)$ or $(\psi - \phi)$ are defined-

Euler Parameters

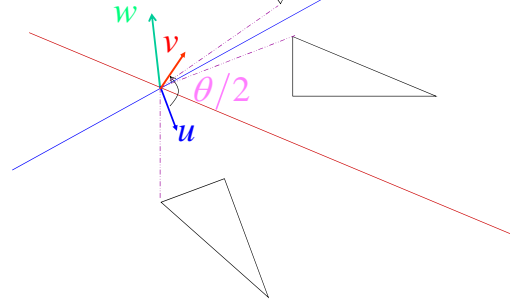
Quaternion

Rotations:

Product of two plane symmetries

$$u \cdot v = \cos \theta/2$$

$$u \times v = w \sin \theta/2$$



Euler 4-Parameters

$$\lambda_0 = \cos \theta/2 ;$$

$$\lambda_1 = w_1 \sin \theta/2 ;$$

$$\lambda_2 = w_2 \sin \theta/2 ;$$

$$\lambda_3 = w_3 \sin \theta/2 ;$$

Normality condition

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

Rotation Matrix

$$R_{0(n+1)}(q) = \begin{pmatrix} 2(\lambda_0^2 + \lambda_1^2) - 1 & 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & 2(\lambda_0^2 + \lambda_2^2) - 1 & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3 - \lambda_0\lambda_2) & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) & 2(\lambda_0^2 + \lambda_3^2) - 1 \end{pmatrix}$$

$$R_{0(n+1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Sign Determination:

$(\lambda_0\lambda_1), (\lambda_0\lambda_2), (\lambda_0\lambda_3), (\lambda_1\lambda_2), (\lambda_1\lambda_3)$ and $(\lambda_2\lambda_3)$

$$\lambda_3 = \frac{\eta}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1} ;$$

$$\lambda_2 = \frac{\eta}{2} \operatorname{sgn}(r_{21} - r_{12}) \sqrt{-r_{11} - r_{22} + r_{33} + 1} ;$$

$$\lambda_1 = \frac{\eta}{2} \operatorname{sgn}(r_{13} - r_{31}) \sqrt{-r_{11} + r_{22} - r_{33} + 1} ;$$

$$\lambda_0 = \frac{\eta}{2} \operatorname{sgn}(r_{32} - r_{23}) \sqrt{r_{11} - r_{22} - r_{33} + 1} ;$$

where $\eta = \pm 1$

Lemma: For all rotations, at least one of the Euler Parameters has a magnitude larger than or equal to 1/2.

Algorithm

	$ \lambda_0(t_{i-1}) $	$ \lambda_1(t_{i-1}) $	$ \lambda_2(t_{i-1}) $	$ \lambda_3(t_{i-1}) $
$\lambda_0(t_i)$	$\Delta_0/4$	$(s_{32} - s_{23})/\Delta_1$	$(s_{13} - s_{31})/\Delta_2$	$(s_{21} - s_{12})/\Delta_3$
$\lambda_1(t_i)$	$(s_{32} - s_{23})/\Delta_0$	$\Delta_1/4$	$(s_{21} + s_{12})/\Delta_2$	$(s_{13} + s_{31})/\Delta_3$
$\lambda_2(t_i)$	$(s_{13} - s_{31})/\Delta_0$	$(s_{21} + s_{12})/\Delta_1$	$\Delta_2/4$	$(s_{32} + s_{23})/\Delta_3$
$\lambda_3(t_i)$	$(s_{21} - s_{12})/\Delta_0$	$(s_{13} + s_{31})/\Delta_1$	$(s_{32} + s_{23})/\Delta_2$	$\Delta_3/4$

with

$$\Delta_0 = 2\operatorname{sgn}(\lambda_0(t_{i-1})) \sqrt{s_{11} + s_{22} + s_{33} + 1} ;$$

$$\Delta_1 = 2\operatorname{sgn}(\lambda_1(t_{i-1})) \sqrt{s_{11} - s_{22} - s_{33} + 1} ;$$

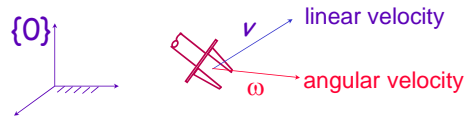
$$\Delta_2 = 2\operatorname{sgn}(\lambda_2(t_{i-1})) \sqrt{-s_{11} + s_{22} - s_{33} + 1} ;$$

$$\Delta_3 = 2\operatorname{sgn}(\lambda_3(t_{i-1})) \sqrt{-s_{11} - s_{22} + s_{33} + 1} .$$

Euler Angles & Parameters

$$\begin{aligned}\lambda_0 &= \cos \theta / 2 \cdot \cos (\psi + \phi) / 2 ; \\ \lambda_1 &= \sin \theta / 2 \cdot \cos (\psi - \phi) / 2 ; \\ \lambda_2 &= \sin \theta / 2 \cdot \sin (\psi - \phi) / 2 ; \\ \lambda_3 &= \cos \theta / 2 \cdot \sin (\psi + \phi) / 2 .\end{aligned}$$

Basic Jacobian



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6 \times 1)} = J_0(q)_{(6 \times n)} \dot{q}_{(n \times 1)}$$

Jacobian for X

Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q) \dot{q}$$

$$J_x(q) = E(x) J_0(q)$$

Basic Jacobian $\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q) \dot{q}$

Jacobian and Basic Jacobian

$$J = \begin{pmatrix} J_{XP} \\ J_{XR} \end{pmatrix} = \begin{pmatrix} E_P & 0 \\ 0 & E_R \end{pmatrix} \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

$$\underline{J(q)} = \underline{E(X)} \underline{J_0(q)}$$

$$\underline{\begin{pmatrix} v \\ \omega \end{pmatrix}} = \underline{J_0(q)} \dot{q}$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_p(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \ y \ z)^T = (\rho \cos \theta \ \rho \sin \theta \ z)^T$

$$E_p(X) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta / \rho & \cos \theta / \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \ y \ z)^T = (\rho \cos \theta \sin \phi \ \rho \sin \theta \sin \phi \ \rho \cos \theta)^T$$

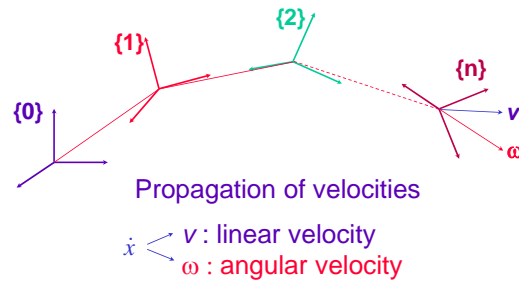
$$E_p(X) = \begin{pmatrix} \cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi \\ -\sin \theta / (\rho \sin \phi) & \cos \theta / (\rho \sin \phi) & 0 \\ \cos \theta \cos \phi / \rho & \sin \theta \cos \phi / \rho & -\sin \phi / \rho \end{pmatrix}$$

Euler Angles

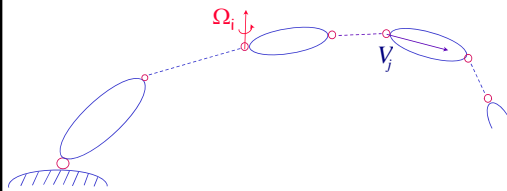
$$x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha.c\beta}{s\beta} & \frac{c\alpha.c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

Singularity of the representation
for $\beta = k\pi$

Spatial Mechanisms



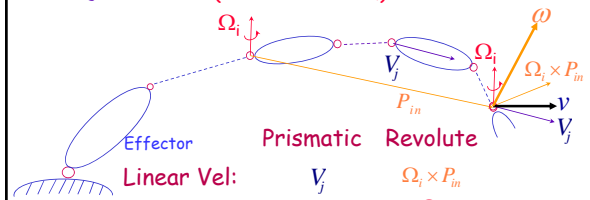
The Jacobian (EXPLICIT FORM)



Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

The Jacobian (EXPLICIT FORM)



Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

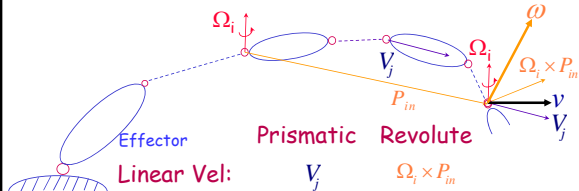
Effector Linear Velocity

$$v = \sum_{i=1}^n [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \iff V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^n \bar{\epsilon}_i \Omega_i \iff \Omega_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity

$$v = \sum_{i=1}^n [\epsilon_i Z_i + \bar{\epsilon}_i (Z_i \times P_{in})] \dot{q}_i \iff V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^n (\bar{\epsilon}_i Z_i) \dot{q}_i \iff \Omega_i = Z_i \dot{q}_i$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{in})] \dot{q}_1 + \dots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1} (Z_{n-1} \times P_{(n-1)n})] \dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{in}) & \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) & \dots \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$v = J_v \dot{q}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \dots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \dots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\omega = J_\omega \dot{q}$$

The Jacobian

$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_p = \frac{\partial x_p}{\partial q_1} \dot{q}_1 + \frac{\partial x_p}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x_p}{\partial q_n} \dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_p}{\partial q_1} & \frac{\partial x_p}{\partial q_2} & \dots & \frac{\partial x_p}{\partial q_n} \end{pmatrix}$$

Jacobian in a Frame

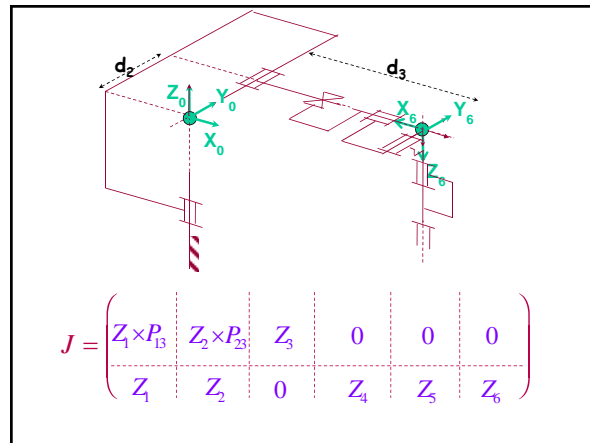
Vector Representation

$$J = \begin{pmatrix} \frac{\partial x_p}{\partial q_1} & \frac{\partial x_p}{\partial q_2} & \dots & \frac{\partial x_p}{\partial q_n} \\ \bar{\epsilon}_1 \cdot Z_1 & \bar{\epsilon}_2 \cdot Z_2 & \dots & \bar{\epsilon}_n \cdot Z_n \end{pmatrix}$$

In $\{0\}$

$${}^0J = \begin{pmatrix} \frac{\partial^0 x_p}{\partial q_1} & \frac{\partial^0 x_p}{\partial q_2} & \dots & \frac{\partial^0 x_p}{\partial q_n} \\ \bar{\epsilon}_1 \cdot {}^0Z_1 & \bar{\epsilon}_2 \cdot {}^0Z_2 & \dots & \bar{\epsilon}_n \cdot {}^0Z_n \end{pmatrix}$$

Stanford Scheinman Arm



Stanford Scheinman Arm Jacobian

$${}^0J = \begin{pmatrix} \frac{\partial^0 x_p}{\partial q_1} & \frac{\partial^0 x_p}{\partial q_2} & \frac{\partial^0 x_p}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix}$$

$$\begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ 1 & 0 & 0 & c_2 & -s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2 \end{bmatrix}$$

Velocity/Force Duality

$$\dot{x} = J \dot{\theta}$$

$$\tau = J^T F$$

Instantaneous Inverse Kinematics



Linearized Kinematic Model

$$\delta x = J(q) \delta q$$

Resolved Motion-Rate

(Whitney 1972)

$$\delta q = J^{-1}(q) \delta x$$

Jacobian

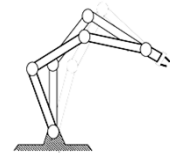
$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = J_{2 \times 2} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix}$$

Inverse Jacobian

$$\begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = J_{2 \times 2}^{-1} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

Redundancy

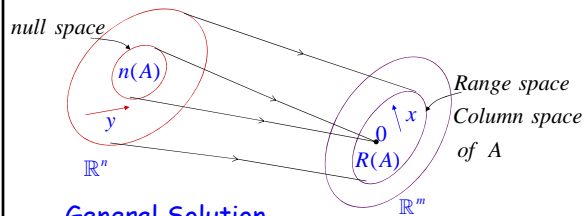
$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = J_{2 \times 3} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix}$$



Generalized Inverse

$$\begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix} = J_{3 \times 2}^{\#} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + [I - J^{\#} J]_{3 \times 3} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix}$$

System $A_{(m \times n)} y_{(n \times 1)} = x_{(m \times 1)}$



$$y = A^{\#} x + [I_n - A^{\#} A] y_0$$

Generalized Inverse

$$A_{(m \times n)} ; \text{rank}(A) = r$$

$$A_{(n \times m)}^{\#} : AA^{\#}A = A$$

Example

$$A = \begin{pmatrix} 2 & -1 \end{pmatrix}$$

$$A^{\#} = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix}$$

Example

$$Ay = (2 \quad -1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x$$

$$A^\# x = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix} x = \begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$Ay = (2 \quad -1) \begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = x$$

$$A_{(m \times n)} y_{(n \times 1)} = x_{(m \times 1)}$$

- $\frac{n > m}{(r = m)} \rightarrow$ Less equations than unknowns
- \rightarrow Free variables
- $\rightarrow \infty$ solutions

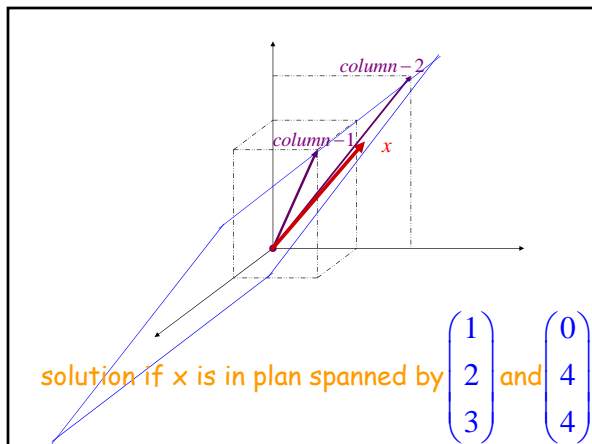
Example $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- $\frac{n < m}{(r = n)} \rightarrow$ More equations than unknowns
- \rightarrow At most one solution

Example $\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

solution if x is in plan spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$



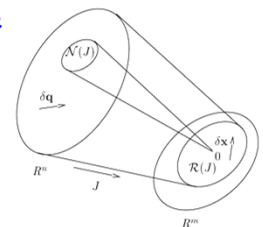
Jacobian Generalized Inverse

Generalized Inverse

$$J_0^\# : J_0 J_0^\# J_0 = J_0$$

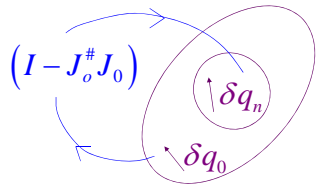
General Solution

$$\delta q = J_0^\# \delta x_0 + [I_n - J_0^\# J_0] \delta q_0$$



General Solution

$$\delta q = J_0^\# \delta x_0 + \underbrace{[I_n - J_0^\# J_0]}_{\delta q_n} \delta q_0$$



$$\delta q_n = (I - J_0^\# J_0) \delta q_0$$

$$0 = J_0 \delta q_n$$

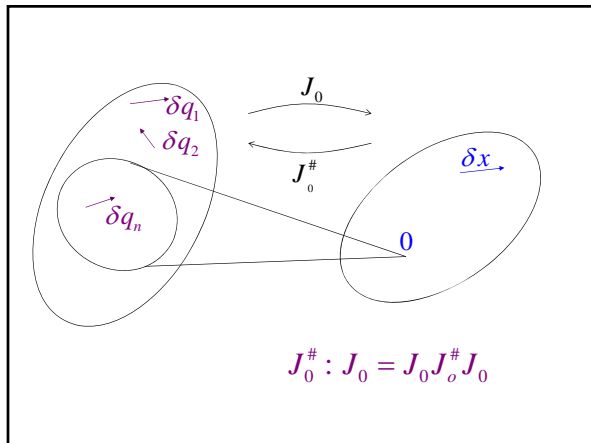
$$\Rightarrow 0 = J_0 (I - J_0^\# J_0) \delta q_0$$

\Rightarrow

$$0 \equiv J_0 - J_0 J_0^\# J_0$$

\Rightarrow

$$J_0^\# : J_0 \triangleq J_0 J_0^\# J_0$$



Pseudo Inverse

$$AA^+A = A$$

$$A^+AA^+ = A^+$$

$$(A^+A)^T = A^+A$$

$$(AA^+)^T = AA^+$$

$A^+ : \text{unique}$

Pseudo-Inverse

Left Inverse

$$m > n \quad A^+ = (A^T A)^{-1} A^T$$

$$(r = n) \quad A^+ A = I$$

$$m = n = r \quad A^+ = A^{-1}$$

$$A^+ A = A A^+ = I$$

Right Inverse

$$m < n \quad A^+ = A^T (A A^T)^{-1}$$

$$(r = m) \quad A A^+ = I$$

Generalized Inverse

Left Inverse

$$m > n \quad A^\# = (A^T W^{-1} A)^{-1} A^T W^{-1}$$

$$(r = n) \quad A^\# A = I$$

$$m = n = r \quad A^\# = A^{-1}$$

$$A^\# A = A A^\# = I$$

Right Inverse

$$m < n \quad A^\# = W^{-1} A^T (A W^{-1} A^T)^{-1}$$

$$(r = m) \quad A A^\# = I$$

Reduction to the Basic Kinematic Model

Initial Problem (m equations)

$$J \delta q = \delta x$$

Reduced Problem (m_0 equations) $J = E J_0$

$$\delta x = E(X) \delta x_0$$

$$J_0(q) \delta q = \delta x_0$$

Solving $\delta x = E(X) \delta x_0$

$E(X)$: $m \times m_0$ matrix ($m \geq m_0$)

$$- \text{rank}(E(X)) \leq m_0$$

- $\text{rank}(E(X)) < m_0$ at configuration x where the representation is singular

Left Inverse

If $\text{rank}(E(X)) = m_0$ the system has a unique solution:

$$\delta x_0 = E_{(m_0 \times m)}^+(X) \delta x$$

E^+ : is such that $E^+ E = I_{m_0}$

$$E^+ = (E^T E)^{-1} E^T$$

and

$$E^+(X) = \begin{pmatrix} E_p^+(Xp) & 0 \\ 0 & E_r^+(Xr) \end{pmatrix}$$

System

$$\delta x_{m \times 1} = E_{m \times m_0} \delta x_{0 m_0 \times 1}$$

$$E_{m_0 \times m}^T \delta x_{m \times 1} = (E^T E)_{m_0 \times m_0} \delta x_{0 m_0 \times 1}$$

$$(E^T E)^{-1} E^T \delta x = \delta x_0$$

$$\delta x_0 = E^+ \delta x$$

$$E^+ = (E^T E)^{-1} E^T$$

$$E^+ E = \underbrace{(E^T E)^{-1} E^T E}_{\text{Left Inverse}} = I$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_p(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \ y \ z)^T = (\rho \cos \theta \ \rho \sin \theta \ z)^T$

$$E_p(X) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta / \rho & \cos \theta / \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_p^+(X) = E_p^{-1}(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

$$E_p^{-1}(X) = \begin{pmatrix} \cos \theta & -\rho \sin \theta & 0 \\ -\sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \ y \ z)^T = (\rho \cos \theta \sin \phi \ \rho \sin \theta \sin \phi \ \rho \cos \theta)^T$$

$$E_p(X) = \begin{pmatrix} \cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi \\ -\sin \theta / (\rho \sin \phi) & \cos \theta / (\rho \sin \phi) & 0 \\ \cos \theta \cos \phi / \rho & \sin \theta \cos \phi / \rho & -\sin \phi / \rho \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

$$E_p^{-1}(X) = \begin{pmatrix} \cos \theta \sin \phi & \rho \sin \theta \sin \phi & \rho \cos \theta \cos \phi \\ -\sin \theta \sin \phi & \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \\ \cos \phi & 0 & -\rho \sin \phi \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}; \quad E_r(x_r) = \begin{pmatrix} -\hat{s}_1 \\ -\hat{s}_2 \\ -\hat{s}_3 \end{pmatrix}$$

$$E_r^+ = (E_r^T E_r)^{-1} E_r^T$$

$$(E_r^T E_r)^{-1} = (\hat{S}_1^T \hat{S}_1 + \hat{S}_2^T \hat{S}_2 + \hat{S}_3^T \hat{S}_3)^{-1}$$

Example

$$S = (S_1 S_2 S_3) = \begin{pmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{z} = \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{bmatrix} = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

$$\hat{S}_1 = \begin{bmatrix} 0 & 0 & S1 \\ 0 & 0 & -C1 \\ -S1 & C1 & 0 \end{bmatrix}$$

$$\hat{S}_1^T \hat{S}_1 = \begin{bmatrix} S_1^2 & -SC1 & 0 \\ -SC1 & C_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad SC1 = S1 \ C1$$

$$\hat{S}_2 = \begin{bmatrix} 0 & 0 & C1 \\ 0 & 0 & S1 \\ -C1 & -S1 & 0 \end{bmatrix}$$

$$\hat{S}_2^T \hat{S}_2 = \begin{bmatrix} C_1^2 & SC1 & 0 \\ SC1 & S_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{S}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{S}_3^T \hat{S}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E^T E = \sum \hat{S}_i^T \hat{S}_i = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I_3$$

$$\forall X_r = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \Rightarrow E^T E = \sum \hat{S}_i^T \hat{S}_i = 2I_3$$

$$(E_r^T E_r)^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \frac{1}{2} I_3$$

$$E_r^+ = (E_r^T E_r)^{-1} E_r^T = \frac{1}{2} E_r^T$$

$$E_r^+ = \frac{1}{2} (-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T)$$

$$E_r^+ = \frac{1}{2} (\hat{S}_1 \hat{S}_2 \hat{S}_3)$$

Angular Velocity

$$x_r = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}; E_r(x_r) = \begin{pmatrix} -\hat{S}_1 \\ -\hat{S}_2 \\ -\hat{S}_3 \end{pmatrix}$$

$$\dot{X}_r = E_r \omega$$

Solution

$$\omega = \frac{1}{2} E^T \dot{X}_r$$

Direction Cosines - Rotation Error

Instantaneous Angular Error

$$x_r = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}; x_{rd} = \begin{bmatrix} S_{1d} \\ S_{2d} \\ S_{3d} \end{bmatrix}$$

$$\delta x_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} - \begin{pmatrix} S1d \\ S2d \\ S3d \end{pmatrix}$$

$$\omega = \frac{1}{2} E^T \dot{X}_r$$

$$\delta \phi = \frac{1}{2} E^T \delta X_r$$

$$\delta X_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} - \begin{pmatrix} S1d \\ S2d \\ S3d \end{pmatrix}$$

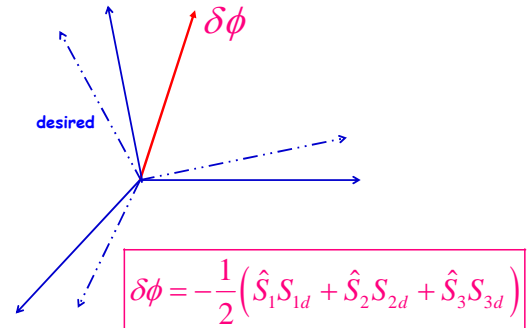
$$E_r^+ = \frac{1}{2} E^T$$

$$E^T = (-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T)$$

$$E_r^+ = \frac{1}{2} (\hat{S}_1 \hat{S}_2 \hat{S}_3)$$

$$E_r^+ \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} = \frac{1}{2} (\hat{S}_1 S1 + \hat{S}_2 S2 + \hat{S}_3 S3) \equiv 0$$

Instantaneous Angular Error



Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}; E_r(x_r) = \begin{pmatrix} -\hat{S}_1 \\ -\hat{S}_2 \\ -\hat{S}_3 \end{pmatrix}$$

Observing $E_r^T(x_r) E_r(x_r) = 2I_3$

$$E_r^+(x_r) = \frac{1}{2} (-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T)$$

Euler Angles

$$E_r(X) = \begin{pmatrix} -S\varphi C\theta/S\theta & C\varphi C\theta/S\theta & 1 \\ C\varphi & S\varphi & 0 \\ S\varphi/S\theta & -C\varphi/S\theta & 0 \end{pmatrix}$$

$$E_r^{-1}(x_r) = \begin{pmatrix} 0 & \cos \psi & \sin \psi \sin \theta \\ 0 & \sin \psi & -\cos \psi \sin \theta \\ 1 & 0 & \cos \theta \end{pmatrix}$$

Euler Parameters

$$x_r = \lambda = (\lambda_0 \lambda_1 \lambda_2 \lambda_3)^T$$

$$\dot{\lambda} = \frac{1}{2} \check{\lambda} \omega$$

$$\check{\lambda} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

$$E_r(\lambda) = \frac{1}{2} \check{\lambda}$$

Euler Parameters

Observing

$$\check{\lambda}^T \check{\lambda} = I_3$$

$$E_r^+(x_r) = 2 \begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

Inverse of the Basic Kinematic Model

System

$$\delta x_{0(m_0 \times 1)} = J_0(q)_{(m_0 \times n)} \delta q_{(n \times 1)}; m_0 \leq n$$

Right Inverse

A solution iff $\text{rank } J_0 = m_0$
 \exists an $n \times m_0$ right inverse $J_0^\# / J_0 J_0^\# = I_{m_0}$

System

$$\delta x_{0(m_0 \times 1)} = J_0(q)_{(m_0 \times n)} \delta q_{(n \times 1)}; m_0 \leq n$$

Solution $\delta q = J_0^\# \delta X_0$

$J_0^\#$: Generalized Inverse

General Solution

$$\delta q = J_0^\# \delta X_0 + \underbrace{[I_n - J_0^\# J_0]}_{\delta q_n} \delta q_0$$

Redundancy (w.r.t a Task)

$$x = l_1 C1 + l_2 C12 + l_3 C123$$

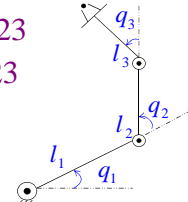
$$y = l_1 S1 + l_2 S12 + l_3 S123$$

$$q_1 = q_2 = 0$$

$$l_1 = l_2 = l_3 = 1$$

$$J_{2 \times 3}(q) = \begin{pmatrix} -S3 & -S3 & -S3 \\ 2+C3 & 1+C3 & C3 \end{pmatrix}$$

$$J_{3 \times 2}^+ = J^T (J J^T)^{-1}$$

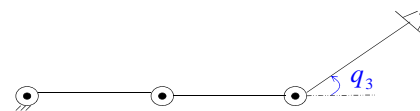


$$J J^T = \begin{pmatrix} 3S_3^2 & -3(1+C3)S3 \\ -3(1+C3)S3 & 3C_3^2 + 6C3 + 5 \end{pmatrix}$$

$$\det(J J^T) = 6S_3^2$$

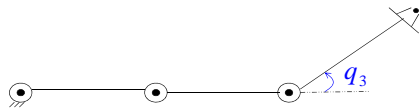
$$(J J^T)^{-1} = \frac{1}{6S_3^2} \begin{pmatrix} 3C_3^2 + 6C3 + 5 & 3(1+C3)S3 \\ 3(1+C3)S3 & 3S_3^2 \end{pmatrix}$$

$$J_{(3 \times 2)}^+ = \frac{1}{6S3} \begin{pmatrix} 1+3C3 & 3S3 \\ -2 & 0 \\ -(5+3C3) & -3S3 \end{pmatrix}$$



$$\delta q = J^+ \delta x + (I_n - J_0^+ J_0) \delta q_0$$

$$J^+ J = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$



$$\delta q = J^+ \delta x + (I_n - J_0^+ J_0) \delta q_0$$

$$(I - J^+ J) = \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\delta q_n = \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \delta q_0$$



$$\delta q_{n1} = \frac{1}{6} (\delta q_1 - 2\delta q_2 + \delta q_3)$$

$$\delta q_{n2} = \frac{1}{6} (-2\delta q_1 + 4\delta q_2 - 2\delta q_3)$$

$$\delta q_{n3} = \frac{1}{6} (\delta q_1 - 2\delta q_2 + \delta q_3)$$

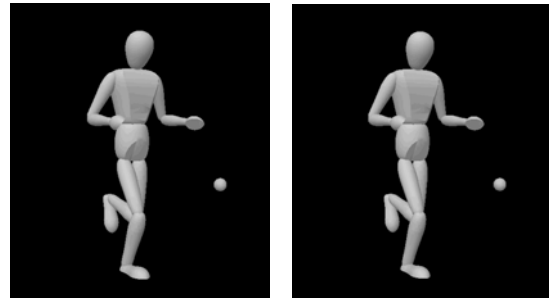
Redundancy

System

$$\delta x_{0(m_0 \times 1)} = J_0(q)_{(m_0 \times n)} \delta q_{(n \times 1)} ; m_0 \leq n$$

General Solution

$$\delta q = J_0^\# \delta X_0 + \underbrace{[I_n - J_0^\# J_0]}_{\delta q_n} \delta q_0$$



Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = (J_1 \ J_2 \ \dots \ J_n)$$

$$\det(J) = 0$$

$$\det({}^i J) = \det({}^j J)$$

Kinematic Singularities

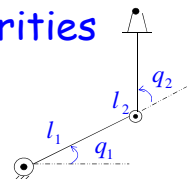
$$x = l_1 C1 + l_2 C12$$

$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$



$$J = S_{01} J_{(1)}$$

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}$$

At Singularity

$$J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

The rank of $(J J^T) < 2$
 $(J^T J) < 2$

Singular Value Decomposition

Theorem - Definition

Any $m \times n$ matrix A of rank r can be factored into:

$$A = U \Sigma V^T; \text{ where}$$

- U is an $m \times m$ orthogonal matrix;
- V is an $n \times n$ orthogonal matrix;

- Σ is an $m \times n$ matrix of the form

$$\Sigma = \left(\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array} \right) ; \text{ with}$$

$\Sigma_r = \text{diag}[\sigma_i]$ with

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

$\sigma_i (i=1, \dots, r)$ are uniquely determined for A and called "Singular values of A "

Decomposition of A

$$A_{(m \times n)} = U_{(m \times m)} \Sigma_{(m \times n)} V^T_{(n \times n)}$$

$$\underline{m \geq n} \quad A^T A = V (\Sigma^T \Sigma) V^T$$

$$\Sigma^T \Sigma = \left(\begin{array}{c|c} \Sigma_r^2 & 0 \\ \hline 0 & 0 \end{array} \right)$$

$$1^\circ) \det(A^T A - \sigma^2 I) = 0 \rightarrow \sigma_1^2, \dots, \sigma_r^2 \rightarrow \Sigma$$

$$2^\circ) (A^T A - \sigma_i^2 I) V_i = 0 \rightarrow V$$

$$3^\circ) AV = U \Sigma$$

$$(A v_1 : A v_2 : \dots) = (\sigma_1 u_1 : \sigma_2 u_2 : \dots)$$

$$u_i = \frac{A v_i}{\sigma_i} \rightarrow U$$

$$\underline{m < n} \quad A A^T = U (\Sigma^T \Sigma) U^T$$

$$(\Sigma \Sigma^T)_{(m \times m)} = \left(\begin{array}{c|c} \Sigma_r^2 & 0 \\ \hline 0 & 0 \end{array} \right)$$

$$1^\circ) \det(AA^T - \sigma^2 I) = 0 \rightarrow \Sigma$$

$$2^\circ) (AA^T - \sigma_i^2 I)U_i = 0 \rightarrow U$$

$$3^\circ) A^T U = V \Sigma$$

$$(A^T u_1 : A^T u_2 : \dots) = (\sigma_1 v_1 : \sigma_2 v_2 : \dots)$$

$$v_i = \frac{A^T u_i}{\sigma_i} \rightarrow V$$

Pseudo Inverse of $A = U \Sigma V^T$ is

$$\underline{A^+ = V \Sigma^+ U^T}$$

$$AA^+A = A$$

$$(A^+A)^T = A^+A$$

$$(AA^+)^T = AA^+$$

Pseudo Inverse of $\Sigma = \begin{pmatrix} \sigma_i & & 0 \\ & \sigma_r & \\ 0 & & \end{pmatrix}$ is

$$\Sigma^+ = \begin{pmatrix} \frac{1}{\sigma_i} & & 0 \\ & \frac{1}{\sigma_i} & \\ 0 & & 0 \end{pmatrix}$$

Example

$$J_{(1)} = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

1°) Σ ?

$$J^T J = \begin{pmatrix} (l_1 + l_2)^2 & (l_1 + l_2)l_2 \\ (l_1 + l_2)l_2 & l_2^2 \end{pmatrix}$$

$$\det(J^T J - \sigma^2 I) = 0$$

$$\sigma_1^2 = l_2^2 + (l_1 + l_2)^2$$

$$\sigma_2^2 = 0$$

$$\Sigma = \begin{pmatrix} \sqrt{l_2^2 + (l_1 + l_2)^2} & 0 \\ 0 & 0 \end{pmatrix}$$

2°) V ?

$$(J^T J - \sigma_1^2 I)v_1 = 0$$

$$\begin{pmatrix} -l_2^2 & (l_1 + l_2)l_2 \\ (l_1 + l_2)l_2 & -(l_1 + l_2)^2 \end{pmatrix} v_1 = 0$$

$$V = \frac{1}{\sqrt{l_2^2 + (l_1 + l_2)^2}} \begin{pmatrix} l_1 + l_2 & -l_2 \\ l_2 & l_1 + l_2 \end{pmatrix}$$

3°) U ?

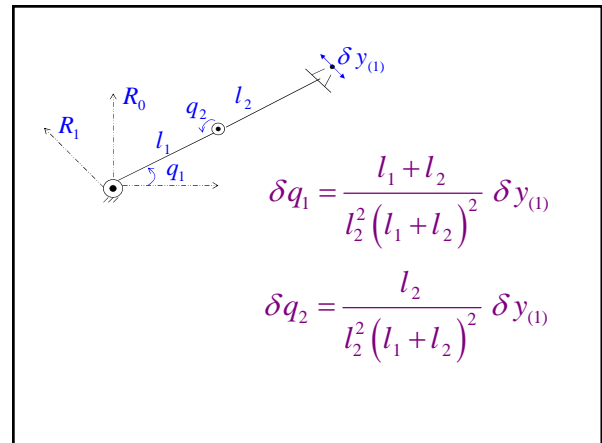
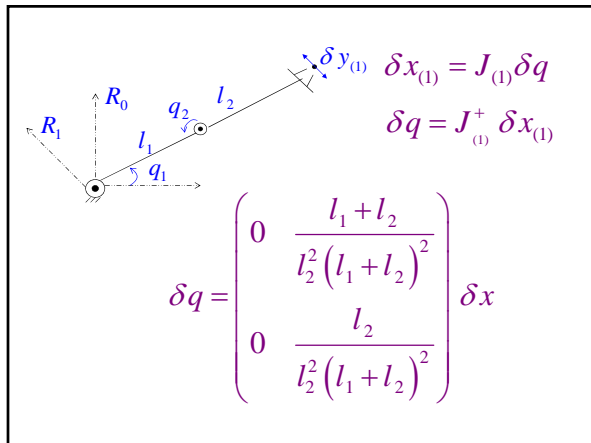
$$u_i = \frac{Jv_i}{\sigma_i}$$

$$u_1 = \begin{pmatrix} 0 & 0 \\ \frac{l_1+l_2}{\sqrt{(\cdot)}} & \frac{l_2}{\sqrt{(\cdot)}} \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2}{\sqrt{(\cdot)}} \\ \frac{l_2}{\sqrt{(\cdot)}} \end{pmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$J = S_{01} U \Sigma V^T$$



$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{l_2^2+(l_1+l_2)^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2}{\sqrt{(\cdot)}} & \frac{l_2}{\sqrt{(\cdot)}} \\ \frac{-l_2}{\sqrt{(\cdot)}} & \frac{l_1+l_2}{\sqrt{(\cdot)}} \end{pmatrix}$$

$$J^+ = V \Sigma^+ U^T S_{01}^T$$

$$J^+ = \begin{pmatrix} 0 & \frac{l_1+l_2}{l_2^2+(l_1+l_2)^2} \\ 0 & \frac{l_2}{l_2^2+(l_1+l_2)^2} \end{pmatrix} \begin{pmatrix} C1 & S1 \\ -S1 & C1 \end{pmatrix}$$

General Solution

$$\delta q = J^+ \delta x + \underbrace{(I - J^+ J)}_{\delta q_n} \delta q_0$$

$$J^+ J = \frac{1}{\sqrt{l_2^2+(l_1+l_2)^2}} \begin{pmatrix} l_1+l_2 & -l_2 \\ l_2 & l_1+l_2 \end{pmatrix}$$

$$\delta q_n = \frac{1}{l_2^2 + (l_1 + l_2)^2} \begin{pmatrix} l_2^2 & -(l_1 + l_2)l_2 \\ -(l_1 + l_2)l_2 & (l_1 + l_2)^2 \end{pmatrix} \delta q_0$$

$$\delta x_n = J \delta q_n = 0$$

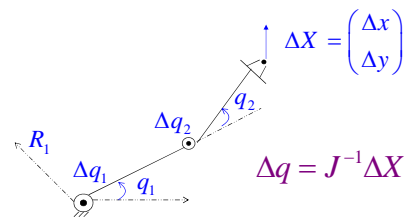
Problem with the Pseudo Inverse Solution

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \underbrace{\begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}}_{J_{(1)}}$$

$$J^{-1} = \frac{1}{l_1 l_2 S2} \underbrace{\begin{pmatrix} l_2 C2 & l_2 S2 \\ -(l_1 + l_2) C2 & -l_2 S2 \end{pmatrix}}_{J_{(1)}^{-1}} \begin{pmatrix} C1 & S1 \\ -S1 & C1 \end{pmatrix}$$

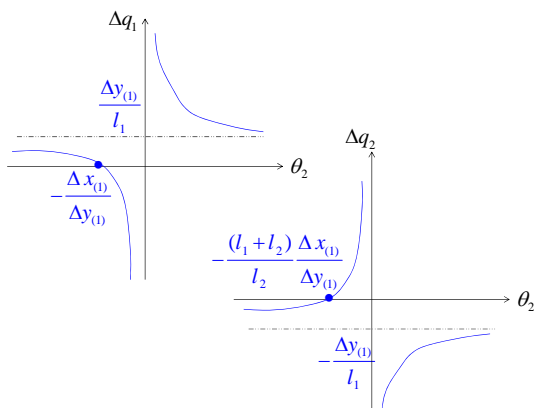
small θ_2

$$J_{(1)}^{-1} \cong \begin{pmatrix} \frac{1}{l_1 \theta_2} & \frac{1}{l_1} \\ -\frac{l_1 + l_2}{l_1 l_2 \theta_2} & -\frac{1}{l_1} \end{pmatrix}$$

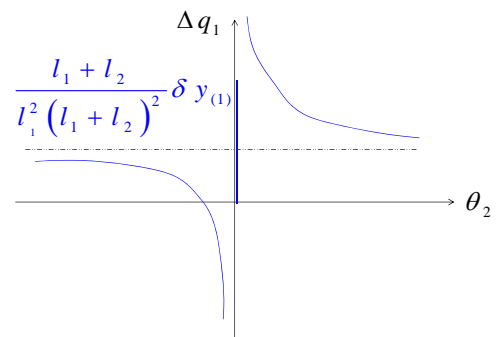


$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{(l_1 + l_2) \Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



Pseudo Inverse Solution



Singularity Robust Inverse

Pseudo-Inverse

$$J^+ = J^T (J J^T)^{-1}$$

S-R Inverse

$$J^* = J^T (J J^T + k I)^{-1}$$

Singularity Robust Inverse

