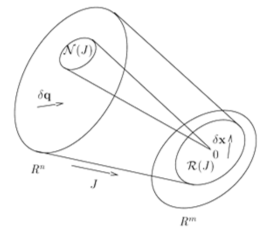
Instantaneous Inverse Kinematics

Jacobian Generalized Inverse

Generalized Inverse

$J_0^{\#}: \quad J_0 J_0^{\#} J_0 = J_0$



General Solution

$$\delta q = J_0^{\#}\,\delta x_0 + \left[I_n - J_0^{\#} J_0\right]\delta q_0$$

$$\delta q_n = \left(I - J_0^{\#} J_0\right)\delta q_0$$

$$0 = J_0\,\delta q_n$$

$$\Rightarrow \quad 0 = J_0\left(I - J_0^{\#} J_0\right)\delta q_0$$

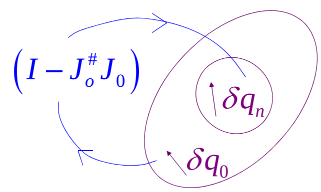
$$0 \equiv J_0 - J_0 J_0^{\#} J_0$$

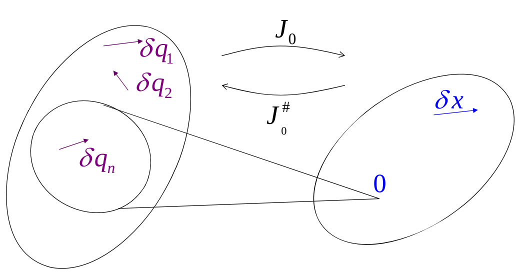
$$\Rightarrow$$

$$J_0^{\#}: \qquad J_0 \triangleq J_0 J_0^{\#} J_0$$

General Solution

$$\delta q = J_0^{\#}\,\delta x_0 + \underbrace{\left[I_n - J_0^{\#} J_0\right]\delta q_0}_{\delta q_n}$$





$$J_0^{\#}: J_0 = J_0 J_o^{\#} J_0$$

Pseudo Inverse

$$A A^{+} A = A$$

$$A^{+} A A^{+} = A^{+}$$

$$\left(A^{+} A\right)^{T} = A^{+} A$$

$$\left(A A^{+}\right)^{T} = A A^{+}$$

A^{+} : *unique*

Pseudo-Inverse

Left Inverse

$m > n$ $\quad A^+ = \left(A^T A\right)^{-1} A^T$

$(r = n)$ $\quad A^+ A = I$

$m = n = r$ $\quad A^+ = A^{-1}$

$A^+ A = A A^+ = I$

Right Inverse

$m < n$ $\quad A^+ = A^T (AA^T)^{-1}$

$(r = m)$ $\quad A A^+ = I$

Generalized Inverse

Left Inverse

$m > n$ $\quad A^\# = \left(A^T W^{-1} A\right)^{-1} A^T W^{-1}$

$(r = n)$ $\quad A^\# A = I$

$m = n = r$ $\quad A^\# = A^{-1}$

$A^\# A = A A^\# = I$

Right Inverse

$m < n$ $\quad A^\# = W^{-1} A^T (AW^{-1} A^T)^{-1}$

$(r = m)$ $\quad AA^\# = I$

Reduction to the Basic Kinematic Model

Initial Problem (m equations)

$$J\,\delta q = \delta x$$

Reduced Problem (m_0 equations) $\quad J = E\,J_0$

$$\delta x = E(X)\,\delta x_0$$

$$J_0(q)\,\delta q = \delta x_0$$

Left Inverse

If $\quad rank\left(E(x)\right) = m_0$
the system has a unique solution:

$$\delta x_0 = E^+_{(m_0 \times m)}(x)\ \delta x$$

E^+ : is such that $\quad E^+ E = \mathrm{I}_{m_0}$

$$E^+ = \left(E^T E\right)^{-1} E^T$$

and

$$E^+(x) = \begin{pmatrix} E_p^+(x_p) & 0 \\ 0 & E_r^+(x_r) \end{pmatrix}$$

Task Jacobian and Basic Jacobian

$$J_x(q) = E(x)\ J_0(q)$$

where

$$E(x) = \begin{pmatrix} E_P(x_p) & 0 \\ 0 & E_R(x_R) \end{pmatrix}$$

Position Representations

$$\dot{x}_P = E_P(x_P)\ v$$

Cartesian Coordinates (x, y, z)

$$E_P(x_P) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \;\; y \;\; z)^T = (\rho\cos\theta \;\; \rho\sin\theta \;\; z)^T$

$$E_P(x) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ,θ,ϕ)

Using

$$(x\;\;y\;\;z)^T = (\rho\cos\theta\sin\phi\quad \rho\sin\theta\sin\phi\quad \rho\cos\theta)^T$$

$$E_P(x_P) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

Position Representations (inverse)

$$v = E_P^{-1}(x)\dot{x}_p$$

Cartesian Coordinates (x, y, z)

$$E_P^{-1}(x_P) = I_3$$

Cylindrical Coordinates (ρ,θ,z)

$$E_P^{-1}(x_P) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ -\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ,θ,ϕ)

$$E_P^{-1}(x_P) = \begin{pmatrix} \cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ -\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix};\; E_r(x_r) = \begin{pmatrix} -\hat{r}_1 \\ -\hat{r}_2 \\ -\hat{r}_3 \end{pmatrix}$$

$$\dot{x}_r = E_r\omega$$

Direction Cosines – Rotation Error

Instantaneous Angular Error

$$x_R = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix};\; x_{Rd} = \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

$$\delta x_R = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$E_r(x_r) = \begin{pmatrix} -\hat{r}_1 \\ -\hat{r}_2 \\ -\hat{r}_3 \end{pmatrix}$$

$$E_r^+ = \left(E_r^T E_r\right)^{-1} E_r^T$$

$$\left(E_r^T E_r\right)^{-1} = \left(\hat{r}_1^T\hat{r}_1 + \hat{r}_2^T\hat{r}_2 + \hat{r}_3^T\hat{r}_3\right)^{-1} = \frac{1}{2}$$

$$\omega = \frac{1}{2} E_R^T \dot{x}_R$$

$$\delta\phi = \frac{1}{2} E_R^T \delta x_R$$

$$\delta x_R = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{pmatrix}$$

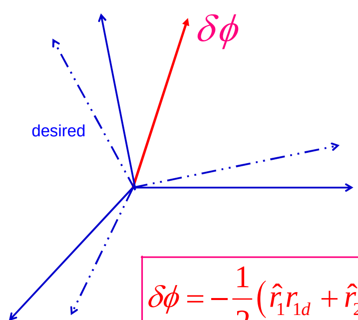
$$E_r^+ = \frac{1}{2} E^T$$

$$E^T = \left(-\hat{r}_1^T - \hat{r}_2^T - \hat{r}_3^T \right)$$

$$E_r^+ = \frac{1}{2} \left(\hat{r}_1 \; \hat{r}_2 \; \hat{r}_3 \right)$$

$$E_r^+ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{2} (\hat{r}_1 r_1 + \hat{r}_2 r_2 + \hat{r}_3 r_3) \equiv 0$$

Instantaneous Angular Error



$$\delta\phi = -\frac{1}{2} \left(\hat{r}_1 r_{1d} + \hat{r}_2 r_{2d} + \hat{r}_3 r_{3d} \right)$$

Euler Angles

$$E_r(X) = \begin{pmatrix} -{}^{S\varphi C\theta}\!/_{S\theta} & {}^{C\varphi C\theta}\!/_{S\theta} & 1 \\ C\varphi & S\varphi & 0 \\ {}^{S\varphi}\!/_{S\theta} & -{}^{C\varphi}\!/_{S\theta} & 0 \end{pmatrix}$$

$$E_r^{-1}(x_r) = \begin{pmatrix} 0 & \cos\psi & \sin\psi \sin\theta \\ 0 & \sin\psi & -\cos\psi \sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}$$

Euler Parameters

$$x_r = \lambda = \left(\lambda_0 \; \lambda_1 \; \lambda_2 \; \lambda_3 \right)^T$$

$$\dot{\lambda} = \frac{1}{2} \check{\lambda} \, \omega$$

$$\check{\lambda} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

$$E_r(\lambda) = \frac{1}{2} \check{\lambda}$$

Euler Parameters

Observing

$$\check{\lambda}^T \check{\lambda} = I_3$$

$$E_r^+(x_r) = 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{(m_0\times n)}\delta q_{(n\times 1)}$$

Solution $\delta q = J_0^{\#}\,\delta X_0$

$J_0^{\#}$: Generalized Inverse

General Solution

$$\delta q = J_0^{\#}\,\delta X_0 + \underbrace{\left[I_n - J_0^{\#} J_0\right]\delta q_0}_{\delta q_n}$$

Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = (J_1\ J_2\ \cdots\ J_n)$$

Singular Value Decomposition

Theorem - Definition

Any m x n matrix A of rank r can be factored into:

$A = U \Sigma V^T$; where

$-U$ is an m x m orthogonal matrix;

$-V$ is an n x n orthogonal matrix;

$-\Sigma$ is an m x n matrix of the form

$$\Sigma = \left(\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right)$$; with

$\Sigma_r = diag[\sigma_i]$ with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

$\sigma_i (i = 1, \cdots, r)$ are uniquely determined for A and called “Singular values of A”

Singularity Robust Inverse

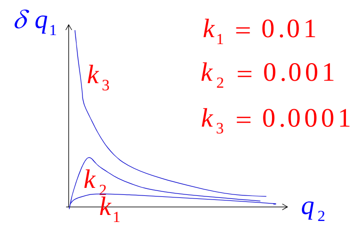
Pseudo-Inverse

$$J^{+} = J^{T}\left(J\,J^{T}\right)^{-1}$$

S-R Inverse

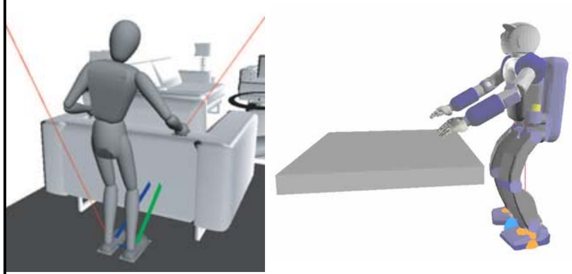
$$J^{*} = J^{T}\left(J\,J^{T} + k\,I\right)^{-1}$$

Singularity Robust Inverse



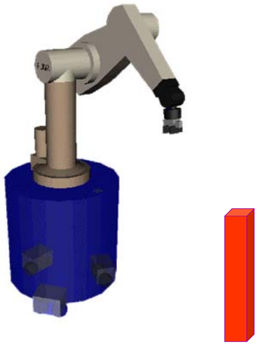
Operational Space Framework!

Whole-body Compliance!

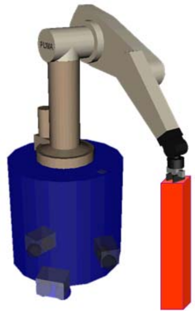


.. motion in contact

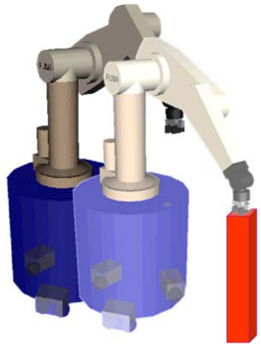
Joint Space Control



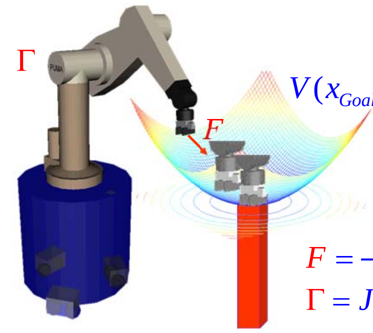
Joint Space Control



Joint Space Control

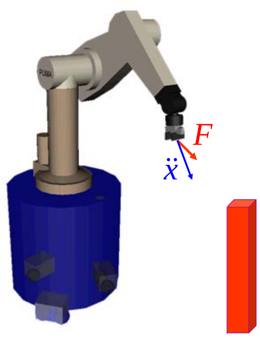


Task-Oriented Control



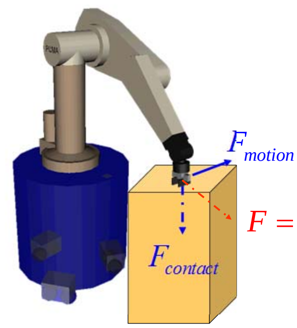
$$F = -\nabla V(x_{Goal})$$
$$\Gamma = J^T F$$

Task-Oriented Dynamics



$$\Lambda \ddot{x} + \mu + p = F$$
$$F = F(\text{dynamics})$$

Unified Motion & Force Control



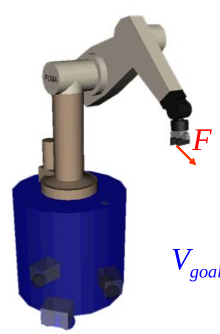
$$F = F_{motion} + F_{contact}$$

Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

with $L(x, \dot{x}) = T(x, \dot{x}) - V(x)$

End-Effector Control



$$\Gamma = J^T(q) F$$
$$F = -\nabla V(x_{Goal})$$
$$V_{goal} = \frac{1}{2} k_p \left(x - x_g\right)^T \left(x - x_g\right)$$

Passive Systems

$$U_{goal} = \frac{1}{2} k_p \left(x - x_g\right)^T \left(x - x_g\right)$$

System $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial (T - V)}{\partial x} = F$

$$\Downarrow \quad F = -\frac{\partial}{\partial X}\left(V_{goal} - \hat{V}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial \left(T - V_{goal}\right)}{\partial x} = 0$$

Stable

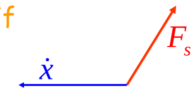
Conservative Forces

Asymptotic Stability

a system $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial \left(T - V_{goal}\right)}{\partial x} = F_s$

is asymptotically stable if

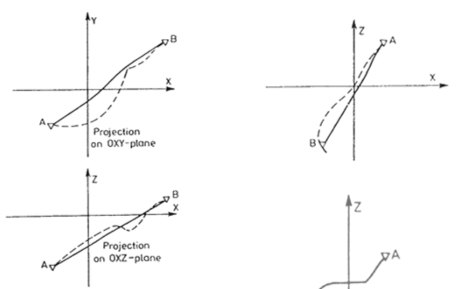
$$F_s^T \dot{x} < 0 \quad ; \; for \; \dot{x} \neq 0$$

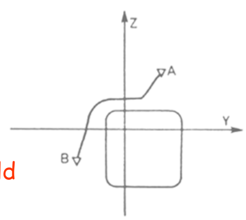


$$F_s = -k_v \dot{x} \quad \rightarrow \quad k_v > 0$$

Control

$$F = -k_p \left(x - x_g\right) - k_v \dot{x} + \hat{p}$$





Artificial Potential Field

Operational Space Dynamics

$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

x: End-Effector Position and Orientation

$\Lambda(x)$: End-Effector Kinetic Energy Matrix

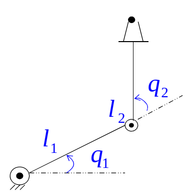
$\mu(x,\dot{x})$: End-Effector Centrifugal and Coriolis forces

$p(x)$: End-Effector Gravity forces

F: End-Effector Generalized forces

Example 2-d.o.f arm

$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$



$$F = -k_p \left(x - x_g\right) - k_v \dot{x} + \hat{p}(x)$$

$$\left(m_1^* c^2 12 + m_2\right)\ddot{x} + m_1^* \ddot{y} + \mu_1 = -k_p \left(x - x_g\right) - k_v \dot{x}$$

$$\left(m_1^* c^2 12 + m_2\right)\ddot{y} + m_1^* \ddot{x} + \mu_2 = -k_p \left(y - y_g\right) - k_v \dot{y}$$

Closed loop behavior

$$m_{11}(q)\ddot{x} + k_v \dot{x} + k_p \left(x - x_g\right) = -\left(m_1^* \ddot{y} + \mu_1\right)$$

$$m_{22}(q)\ddot{y} + k_v \dot{y} + k_p \left(y - y_g\right) = -\left(m_1^* \ddot{x} + \mu_2\right)$$

Joint Space Dynamics

$$A(q)\ddot{q} + b(q,\dot{q}) + g(q) = \Gamma$$

q: Joint Coordinates

$A(q)$: Kinetic Energy Matrix

$b(q,\dot{q})$: Centrifugal and Coriolis forces

$g(q)$: Gravity forces

Γ: Generalized forces

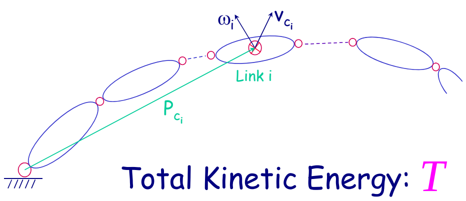
Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

$$A(q)\ddot{q} + b(q,\dot{q}) + g(q) = \Gamma$$

$$A(q): T = \frac{1}{2}\dot{q}^T A\,\dot{q} \qquad A(q) \;\Rightarrow\; b(q,\dot{q})$$

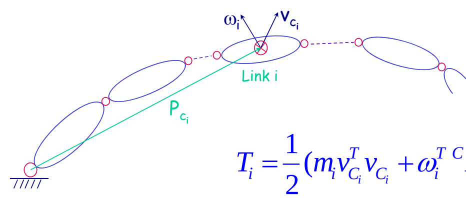
Equations of Motion



Total Kinetic Energy: T

$$T = \sum T_{Link\,i} \equiv \frac{1}{2}\dot{q}^T A\,\dot{q}$$

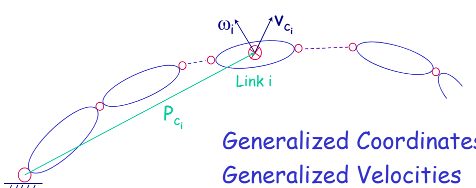
Equations of Motion — Explicit Form



$$T_i = \frac{1}{2}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i)$$

Total Kinetic Energy ⇨ $T = \sum_{i=1}^{n} T_i$

Equations of Motion — Explicit Form



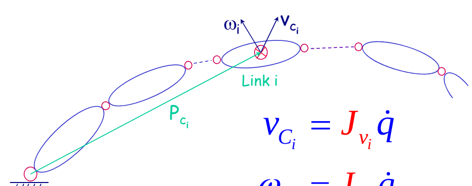
Generalized Coordinates q

Generalized Velocities $\dot{q}$

Kinetic Energy Quadratic Form of Generalized Velocities $T = \frac{1}{2}\dot{q}^T A\,\dot{q}$

$$\frac{1}{2}\dot{q}^T A\,\dot{q} \equiv \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i)$$

Equations of Motion — Explicit Form



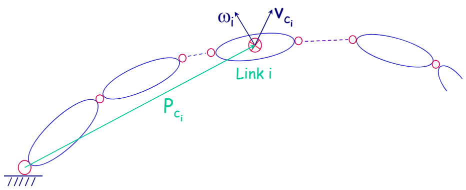
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T A\,\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T\, {}^C I_i J_{\omega_i}\dot{q})$$

Equations of Motion — Explicit Form



$$\frac{1}{2}\dot{q}^T A\,\dot{q} = \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\,{}^C I_i J_{\omega_i})\right]\dot{q}$$

$$A = \sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\,{}^C I_i J_{\omega_i})$$

$$\underset{(n\times n)}{A(q)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Christoffel Symbols

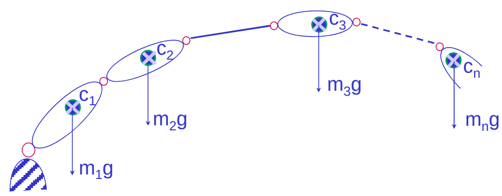
$$b_{ijk} = \frac{1}{2}(a_{ijk} + a_{ikj} - a_{jki}) \qquad a_{ijk} = \frac{\partial a_{ij}}{\partial q_k}$$

$$b(q,\dot{q}) = C(q)\left[\dot{q}^2\right] + B(q)\left[\dot{q}\dot{q}\right]$$

$$\underset{(n\times n)}{C(\mathbf{q})}\underset{(n\times 1)}{[\dot{\mathbf{q}}^2]} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

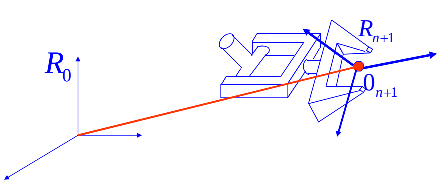
$$\underset{(n\times\frac{(n-1)n}{2})}{B(\mathbf{q})}\ \underset{(\frac{(n-1)n}{2}\times 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix}\begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)}\dot{q}_n \end{bmatrix}$$

Gravity Vector



$$g = -(J_{v_1}^T(m_1 g) + J_{v_2}^T(m_2 g) + \cdots + J_{v_n}^T(m_n g))$$

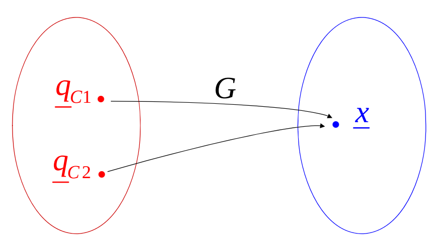
Effector Equations of Motion



Non-Redundant Manipulator ; $n = m_0$

$$x = (x_1\, x_2 \ldots x_{m_0})^T$$
$$q = (q_1\, q_2 \ldots q_n)^T$$
$$x = G(q)$$

Domain



$$D_q = \prod_{i=1}^{n}\left[\underline{q}_i, \overline{q}_i\right] \qquad D_x = G\left(D_q\right)$$

$$\tilde{D}_x = G\left(\tilde{D}_q\right)$$

$\tilde{D}_q$: Excluding Singularities and such that G is one-to-one

In $\tilde{D}_x$, $x_1, x_2, \ldots x_{m_0}$ form a complete set of configuration parameters for the manipulator.

$x_1, \ldots, x_{m_0}$: system of generalized coordinates

Kinetic Energy

$$T(x,\dot{x}) = \frac{1}{2}\dot{x}^T \Lambda(x)\,\dot{x}$$

$\Lambda_{m_0 \times m_0}(x)$: Kinetic Energy Matrix

$$T_x(x,\dot{x}) = \frac{1}{2}\dot{x}^T \Lambda(x)\,\dot{x}$$

Identity

$$T_x(x,\dot{x}) \equiv T_q(q,\dot{q})$$

$$\frac{1}{2}\dot{x}^T \Lambda(X)\dot{x} \equiv \frac{1}{2}\dot{q}^T A(q)\,\dot{q}$$

$$\dot{x} = J\dot{q}$$

$$\Lambda(x) = J^{-T}(q)\; A(q)\; J^{-1}(q)$$

$$p(x) = J^{-T} g(q)$$

System $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial\left(T-U\right)}{\partial x} = F$

$$\frac{\partial T}{\partial \dot{x}} = \frac{1}{2}\frac{\partial}{\partial \dot{x}}\left(\dot{x}^T \Lambda\,\dot{x}\right) = \Lambda\,\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \dot{\Lambda}\dot{x} + \Lambda\ddot{x}$$

$$\frac{\partial T}{\partial x_i} = \frac{1}{2}\dot{x}^T \Lambda_{x_i}\dot{x}$$

$$\mu\left(x,\dot{x}\right) = \dot{\Lambda}\dot{x} - \begin{vmatrix} \frac{1}{2}\dot{x}^T \Lambda_{x_1}\dot{x} \\ \vdots \\ \frac{1}{2}\dot{x}\Lambda_{x_{m_0}}\dot{x} \end{vmatrix}$$

$$\mu(x,\dot{x}) = \dot{\Lambda}\dot{x} - m(x,\dot{x})$$

$$m_i = \frac{1}{2}\dot{x}^T \Lambda_{x_i}\dot{x} \qquad ; \quad \Lambda = J^{-T} A J^{-1}$$

$$\begin{cases} \dot{\Lambda}\dot{x} = J^{-T}\dot{A}\dot{q} - \Lambda h(q,\dot{q}) + \dot{J}^{-T} A\dot{q} \\ m(x,\dot{x}) = J^{-T} l(q,\dot{q}) + \dot{J}^{-T} A\dot{q} \end{cases}$$

where $h \triangleq \dot{J}\dot{q}$

$$l_i \triangleq \frac{1}{2}\dot{q}^T A_{q_i}\dot{q}$$

$$m_i(x,\dot{x}) = \frac{1}{2}\dot{x}^T \Lambda_{x_i} \dot{x}$$

$$m(x,\dot{x}) = \frac{1}{2}|\dot{x}^T J^{-T} A_{x_i} J^{-1}\dot{x}| + 2\frac{1}{2}|\dot{x}^T J_{x_i}^{-T} A J^{-1}\dot{x}|$$

$$m(x,\dot{x}) = \frac{1}{2}|\dot{q}^T A_{x_i} \dot{q}| + \dot{J}^{-T} A \dot{q}$$

$$\dot{q}^T A_{x_i}\dot{q} = (\frac{\partial q_1}{\partial x_i}\frac{\partial q_2}{\partial x_i}\cdots\frac{\partial q_n}{\partial x_i}) . |\dot{q}^T A_{q_i} \dot{q}|$$

$$\frac{1}{2}|\dot{q}^T A_{x_i} \dot{q}| = J^{-T} l(q,\dot{q})$$

$$m(x,\dot{x}) = J^{-T} l(q,\dot{q}) + \dot{J}^{-T} A \dot{q}$$

$$\mu = J^{-T}\left(\dot{A}\dot{q} - l(q,\dot{q})\right) - \Lambda h(q,\dot{q})$$

where $h \triangleq \dot{J}\dot{q}$

$$l_i \triangleq \frac{1}{2}\dot{q}^T A_{q_i}\dot{q}$$

$$\mu = J^{-T}(q)\, b(q,\dot{q}) - \Lambda h(q,\dot{q})$$

Joint Space/Operational Space Relationships

$$T_x(x,\dot{x}) \equiv T_q(q,\dot{q})$$

$$\frac{1}{2}\dot{x}^T \Lambda(X)\dot{x} = \frac{1}{2}\dot{q}^T A(q)\dot{q}$$

Using $\dot{x} = J(q)\dot{q}$

$$\frac{1}{2}\dot{q}^T J^T \Lambda\, J\, \dot{q} = \frac{1}{2}\dot{q}^T A\, \dot{q}$$

Joint Space/Operational Space Relationships

$$\Lambda(x) = J^{-T}(q)\, A(q)\, J^{-1}(q)$$

$$\mu(x,\dot{x}) = J^{-T}(q)\, b(q,\dot{q}) - \Lambda(q)\, h(q,\dot{q})$$

$$p(x) = J^{-T}(q)\, g(q)$$

where $h(q,\dot{q}) \doteq \dot{J}(q)\dot{q}$

Λ, μ, and P are all expressed in terms of joint coordinates

The domain $\tilde{D}_x$ can be extended to

$$\bar{D}_x = G\left(\bar{D}_q\right)$$

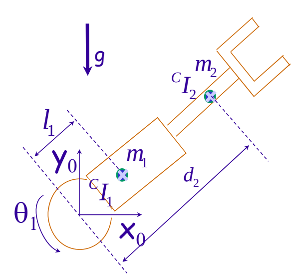
$\bar{D}_q$: domain D_q excluding singularities

Example

$q_2 = d_2$

$$x = \begin{bmatrix} d_2 c1 \\ d_2 s1 \end{bmatrix}$$

$${}^0J = \begin{bmatrix} -d_2 s1 & c1 \\ d_2 c1 & s1 \end{bmatrix}$$



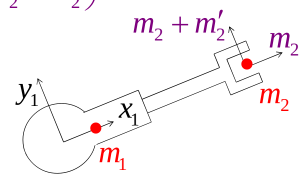
$${}^0J = \begin{bmatrix} -d_2 s1 & c1 \\ d_2 c1 & s1 \end{bmatrix}$$

$${}^0J = \begin{pmatrix} c1 & -s1 \\ s1 & c1 \end{pmatrix} \overbrace{\begin{pmatrix} 0 & 1 \\ d_2 & 0 \end{pmatrix}}^{{}^1J}$$

$${}^1J^{-1} = \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix};$$

$${}^1\Lambda = \begin{pmatrix} 0 & 1 \\ 1/d_2 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix}$$

$${}^1\Lambda = \begin{pmatrix} m_2 & 0 \\ 0 & m_2 + m_2' \end{pmatrix}$$

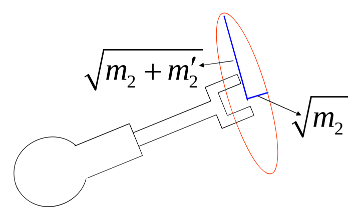


$$m_2' = \frac{I_{221} + I_{222} + m_1 l_1^{\,2}}{d_2^{\,2}}$$

$${}^0\Lambda = \begin{pmatrix} c1 & -s1 \\ s1 & c1 \end{pmatrix} \begin{pmatrix} m_2 & 0 \\ 0 & m_2^{\,+} \end{pmatrix} \begin{pmatrix} c1 & s1 \\ -s1 & c1 \end{pmatrix}$$

$$m_2^{\,+} = m_2 + m_2'$$

$${}^0\Lambda = \begin{pmatrix} m_2 + m_2'\, s1^2 & -m_2' s\,c1 \\ -m_2' s\,c1 & m_2 + m_2'\, c1^2 \end{pmatrix}$$



$${}^0\Lambda = \begin{pmatrix} m_2 + m_2'\, s1^2 & -m_2' s\,c1 \\ -m_2' s\,c1 & m_2 + m_2'\, c1^2 \end{pmatrix}$$

Nonlinear Dynamic Decoupling

Model

$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

Control Structure

$$F = \hat{\Lambda}(x)F^* + \hat{\mu}(x,\dot{x}) + \hat{p}(x)$$

Decoupled System

$$I\,\ddot{x} = F^*$$

with $\Gamma = J^T F$

Dynamic Decoupling

$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

$$F = \hat{\Lambda}F^* + \hat{\mu}(x,\dot{x}) + \hat{p}(x)$$

$$I_{m_0}\ddot{X} = \underbrace{\left(\Lambda^{-1}\hat{\Lambda}\right)}_{G(x)}F^* + \Lambda^{-1}\underbrace{\left(\hat{\mu} - \mu\right)}_{\tilde{\mu}_{(x,\dot{x})}} + \Lambda^{-1}\underbrace{\left(P - \hat{P}\right)}_{\tilde{P}(x)}$$

$$I_{m_0}\ddot{x} = G(x)F^* + \varepsilon(x,\dot{x}) + d(t)$$

$$G(x) = \Lambda^{-1}\hat{\Lambda} \simeq I + \varepsilon_{\Lambda}$$

$$\varepsilon(x,\dot{x}) = \Lambda^{-1}\left(\tilde{\mu} + \tilde{P}\right)$$

$d(t)$: unmodeled disturbances

Perfect Estimates

$$I_{m_0}\ddot{x} = F^*$$

F^* input of decoupled end-effector

Goal Position Control

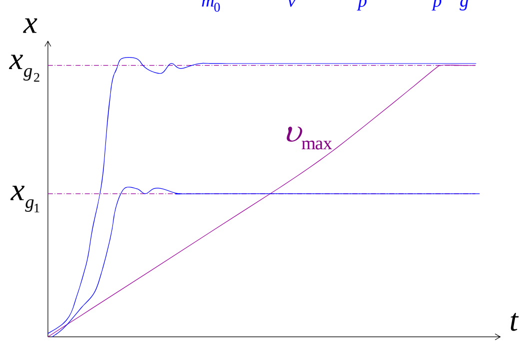
$$F^* = -k_v\dot{x} - k_p\left(x - x_g\right)$$

Closed Loop

$$I_{m_0}\ddot{x} + k_v\dot{x} + k_p x = k_p x_g$$

Closed Loop

$$I_{m_0}\ddot{x} + k_v\dot{x} + k_p x = k_p x_g$$

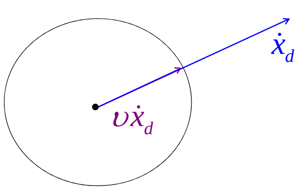


PD Control

$$F^* = -k_v\dot{x} - k_p\left(x - x_g\right)$$

Velocity-Like Control

$$F^* = -k_v\left(\dot{x} - \frac{k_p}{k_v}\left(x_g - x\right)\right)$$



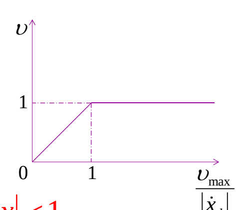
$$\dot{x}_d = \frac{k_p}{k_v}\left(x_g - x\right)$$

$$F^* = -k_v\left(\dot{x} - \underbrace{\frac{k_p}{k_v}\left(x_g - x\right)}_{\dot{x}_d}\right)$$

$$F^* = -k_v\left(\dot{x} - \upsilon\dot{x}_d\right)$$

with

$$\upsilon = sat\left(\frac{V_{max}}{|\dot{x}_d|}\right)$$



$$sat(x) = \begin{cases} x & if\ |x| < 1 \\ sign(x) & if\ |x| > 1 \end{cases}$$

Trajectory Tracking

Trajectory: x_d , $\dot{x}_d$, $\ddot{x}_d$

$$F^* = I_{m_0}\ddot{x}_d - k_v(\dot{x} - \dot{x}_d) - k_p(x - x_d)$$

$$(\ddot{x} - \ddot{x}_d) + k_v(\dot{x} - \dot{x}_d) + k_p(x - x_d)$$

or $$\ddot{\varepsilon}_X + k_v\dot{\varepsilon}_X + k_p\varepsilon_X = 0$$

with $\varepsilon_X = x - x_d$

In joint space

$$\ddot{\varepsilon}_q + k_v\dot{\varepsilon}_q + k_p\varepsilon_q = 0$$

with $\varepsilon_q = q - q_d$