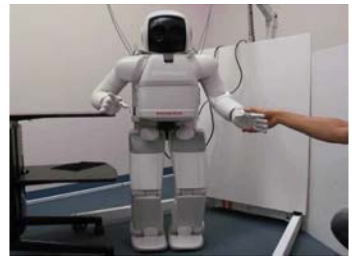
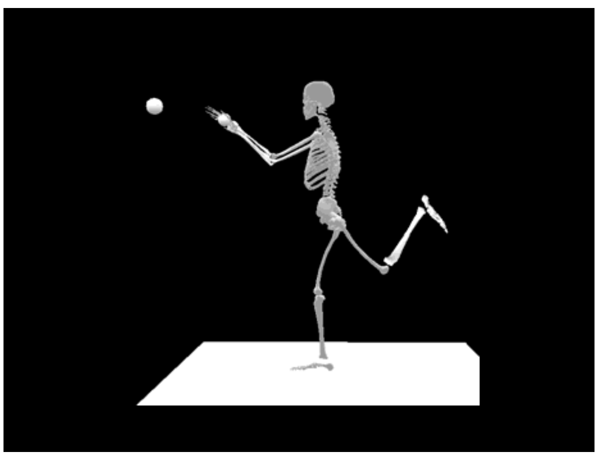
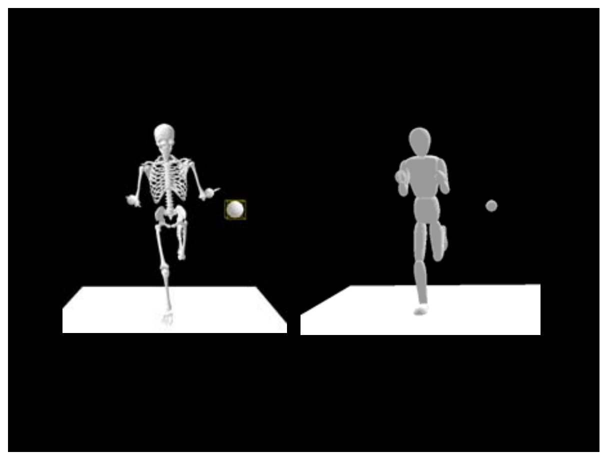
# Redundancy



# Redundancy





## Equations of Motion

Joint Space

$$A(q)\ddot{q} + b(q,\dot{q}) + g(q) = \Gamma$$

Operational Space

$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

Relationships

$$\Gamma = J^T F$$

$$(A\ddot{q} + b + g) = J^T(\Lambda\ddot{x} + \mu + p)$$

$$(A\ddot{q} + b) = J^T(\Lambda\ddot{x} + \mu)$$ Inertial forces

## Non Redundancy

$$A\ddot{q} + b + g = \Gamma$$ (joint dynamics)

$J^{-T}$    $J^T$

$$\Lambda\ddot{x} + \mu + p = F$$ (Task dynamics)

## Redundancy

$$A\ddot{q} + b + g = \Gamma$$ (joint dynamics)

projection $\bar{J}^T$   $J^T$

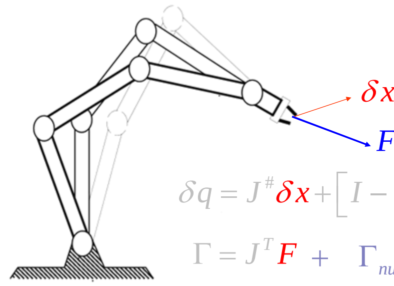
$$\Lambda\ddot{x} + \mu + p = F$$ (Task dynamics)

where

$$\bar{J} = A^{-1} J^T \Lambda \quad \text{and} \quad \Lambda^{-1} = J A^{-1} J^T$$

$\bar{J}$ : dynamically consistent generalized inverse

## Redundancy




$$\delta q = J^{\#}\delta x + \left[I - J^{\#} J\right]\delta q_0$$

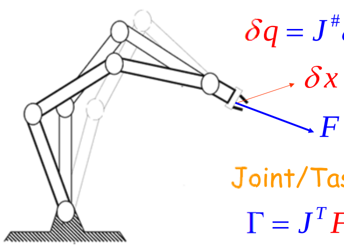
$$\Gamma = J^T F + \Gamma_{nullspace}$$

## Redundancy

Joint/Task Displacements

$$\delta x = J\delta q$$

$$\delta q = J^{\#}\delta x + \left[I - J^{\#} J\right]\delta q_0$$




Joint/Task Forces

$$\Gamma = J^T F$$

$$F = ?$$

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$$\Gamma = J^T F$$

Given $F$ , $\Gamma$ is $\left(J^T F\right)$

Given $\Gamma$ , what is $F$

$$F = J^{\#^T}\Gamma \ ?$$

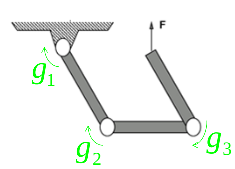
## However

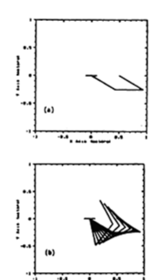
different selections of $J^{\#}$   $\left(J = J\, J^{\#} J\right)$

would lead

to different solutions

## Gravity Example

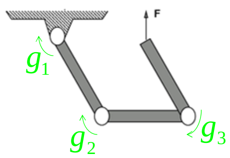


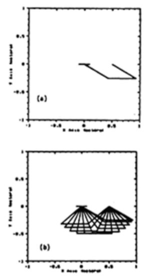



$p = ?$   $p = J^{+^T} g$

$$J^{+} = J^T \left(J\, J^T\right)^{-1}$$

## Gravity Example

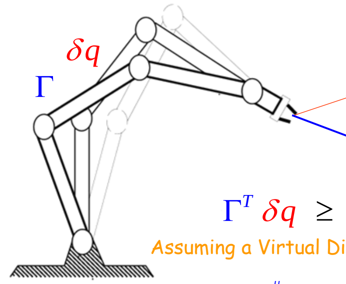


$p = ?$ $\quad p = \bar{J}^T g$

$$\bar{J} = A^{-1} J^T \Lambda$$

## Redundancy




$$\Gamma^T \delta q \geq F^T \delta x$$

Assuming a Virtual Displacement

$$\delta q = J^{\#} \delta x + \left(I - J^{\#} J\right) \delta q_0$$

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Virtual Displacement $\quad \delta q = J^{\#} \delta x + \left(I - J^{\#} J\right) \delta q_0$

Virtual Work $\quad \delta w = \Gamma^T \delta q$

$$\delta w = \delta w_1 + \delta w_2$$

$$\left(J^{\#^T} \Gamma\right)^T \delta x \qquad \left[\left(I - J^{\#} J\right)^T \Gamma\right]^T \delta q_0$$

$$\Downarrow$$

$$\Gamma = J^T (J^{\#^T} \Gamma) + \left(I - J^T J^{\#^T}\right) \Gamma$$

## Decomposition

$$\Gamma = J^T \left[J^{\#^T} \Gamma\right] + \left[I - J^T J^{\#^T}\right] \Gamma$$

Task Space Forces (F)

Joint Torques acting in the null space

$$\Gamma = J^T F + \left[I - J^T J^{\#^T}\right] \Gamma_0$$

## Dynamic Constraints

$$A(q)\, \ddot{q} + b(q, \dot{q}) + g(q) = \Gamma$$

$$\Gamma = J^T F + \left[I - J^T J^{\#^T}\right] \Gamma_0$$

$$A \ddot{q} + (b + g) = J^T F + \left[I - J^T J^{\#T}\right] \Gamma_0$$

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$$\ddot{q} + A^{-1}(b + g) = A^{-1} J^T F + A^{-1}\left[I - J^T J^{\#T}\right] \Gamma_0$$

$$J \; \Downarrow$$

$$J\ddot{q} + JA^{-1}(b + g) = JA^{-1} J^T F + JA^{-1}\left[I - J^T J^{\#T}\right] \Gamma_0$$

$$J\ddot{q} = \ddot{x} - \dot{J}\dot{q} \quad \Downarrow$$

$$\ddot{x} + \left[JA^{-1}(b + g) - \dot{J}\dot{q}\right] = \left(JA^{-1} J^T\right) F + JA^{-1}\left[I - J^T J^{\#T}\right] \Gamma_0$$

$$\Lambda^{-1} \qquad \ddot{x}_n = 0$$

## Dynamic Consistency

$\Gamma \longrightarrow J A^{-1} \Gamma$

joint torques — task acceleration

Relationship

$$\Gamma = J^T F + \left(I - J^T J^{\#^T}\right)\Gamma_0$$

Dynamic Constraint

$$J A^{-1}\left(I - J^T J^{\#^T}\right)\Gamma_0 \equiv 0$$

$$J A^{-1} = \left(J A^{-1} J^T\right) J^{\#^T}$$

$$\Lambda \nwarrow \left(J A^{-1} J^T\right)^{-1} J A^{-1} = J^{\#^T}$$

## Dynamic Consistency

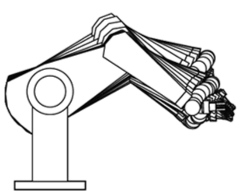
$\bar{J}(q)$ is the Dynamically Consistent Generalized Inverse

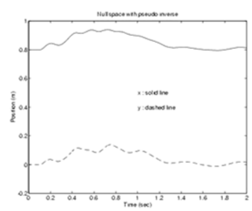
**Theorem (Consistency)**

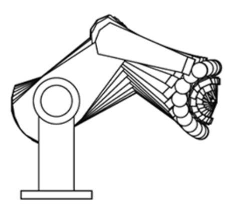
$\bar{J}$ is unique and $\bar{J} = A^{-1} J^T \Lambda$

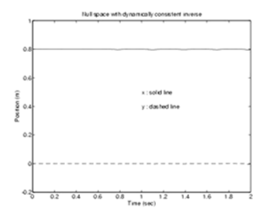
Non-redundant

$$\bar{J} = J^{-1}$$








## Velocity Force Duality

| | Velocity | Force |
|---|---|---|
| Non Red. | $\delta q = J^{-1}\delta x$ | $\Gamma = J^T F$ |
| Redundant | $\delta q = \bar{J}\delta x + \left[I - \bar{J}J\right]\delta q_0$ | $\Gamma = J^T F + \left[I - J^T \bar{J}^T\right]\Gamma_0$ |

## Task dynamics

$$\Lambda(q)\ddot{x} + \mu(q,\dot{q}) + p(q) = F$$

$$\Lambda = \left(J A^{-1} J^T\right)^{-1}$$

$$\mu(q,\dot{q}) = \bar{J}^T b(q,\dot{q}) - \Lambda(q)\dot{J}(q)\dot{q}$$

$$p(q) = \bar{J}^T g(q)$$

## Redundant Robot Control

Task Space: $J^T$

Null Space: $N^T$ where $N = I - \bar{J}J$

Robot Control

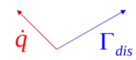
$$\Gamma = J^T F + N^T \Gamma_0$$

$\Gamma_1$ $\Gamma_2$

dynamically decoupled

## Stability

$\Gamma_{dis}^T \dot{q} \le 0$ ; for $\dot{q} \ne 0$




$$\Gamma_{dis} = -k_v J^T \dot{x} = -k_v J^T J \dot{q}$$

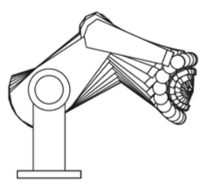
$$\dot{q}^T D(q) \dot{q} \ge 0 \; ; \qquad \dot{q} \ne 0$$

$$D(q) = k_v \left( J^T J \right)$$

$J^T J$ : is a n x n matrix of rank $m_0$
it is Positive Semi-definite

The System is Stable, but not asymptotically stable

$$\dot{q}^T D(q) \dot{q} = 0$$



## Asymptotic Stability

$\Gamma_{dis}^T \dot{q} < 0$ ; for $\dot{q} \ne 0$

$$\Gamma_{dis} = -k_v J^T J \dot{q} - k_v N^T \dot{q}$$

$$\Downarrow$$

$$D(q) = k_v \left( J^T J + N^T \right)$$

Positive definite

$\dot{q}^T D(q) \dot{q} < 0$ for $\dot{q} \ne 0$

## Asymptotic Stability

$\Gamma_{dis}^T \dot{q} < 0$ ; for $\dot{q} \ne 0$

$$\Gamma_{dis} = -k_v J^T J \dot{q} - k_{vq} \dot{q}$$

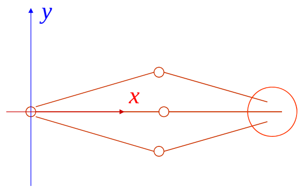
$$\Downarrow$$

$$D(q) = k_v J^T J + k_{vq} I_n$$

Positive definite

$\dot{q}^T D(q) \dot{q} > 0$ for $\dot{q} \ne 0$

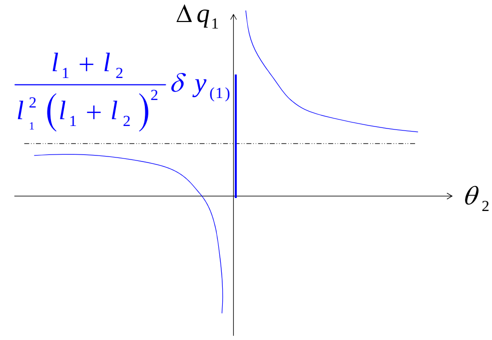
## Kinematic Singularities




Joint Space Formulation

Find a pseudo-inverse $J^+$

## Pseudo Inverse Solution




## Kinematic Singularities

The end-effector **mobility** locally decreases

Singularities

$$S(q) = \det[J(q)] = S_1(q).S_2(q)\cdots S_{n_S}(q)$$

Singular direction

$S_i = 0$ ; $\zeta_i$ → Infinite effective mass → infinite effective inertia

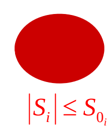
## Kinematic Singularities

Singularity Neighborhood

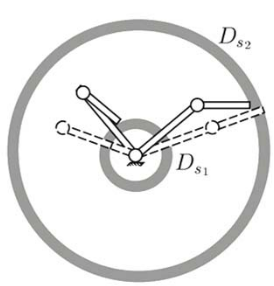
$$S(q) = S_1(q).S_2(q).S_3(q)\cdots S_{n_s}(q)$$

Singularity $S_i$

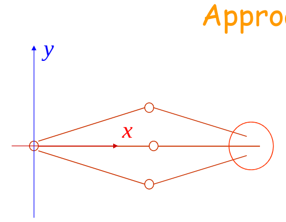
$$D_{S_i} = \{q \mid |S_i(q)| \le S_{0_i}\}$$




Singularity Neighborhood




## Approach




In $D_S$, the robot is treated as redundant w.r.t. motions in the subspace ⊥ to the singular direction

- Along Singular Directions:
  Control in Null Space $\Gamma_{null-space}$
- In subspace ⊥ to singular direction
  Control in sub-O-Space $F_{sub-os}$

## Types of Singularities

Elbow Lock — Wrist Lock — Overhead Lock

Type 1 — Type 2

## Types of Singularities

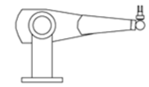
Control Strategy

Type 1

Motion in Null Space

$\Rightarrow$ Motion along/about $\zeta_i$

Control $S_i$

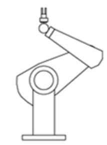


Type 2

Motion in Null Space

$\Rightarrow$ Only changes of $\zeta_i$

Control $\zeta_i$



## Singularity Control

$$\Gamma = J^T_{sub} F_{sub} + N^T_{sub} \Gamma_{s_i}$$

where

$N_{sub} = I - \bar{J}_{sub} J_{sub}$ and $\Gamma_{S_i} = -\nabla V_i(S_i)$

Moving to a singularity

Control $S_i(q)$ to reach $S_i = 0$

Moving out of a singularity

Control $\dot{S}_i$ from zero to the desired Velocity at the singularity boundary